# Notes to the 4-Distance Problem

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## 1 Basic Discussion

**Problem 1.** The *4-diatance problem* is an open Diophantine problem that asks if there exists a point $P(x, y)$ on the 2-dimensional Euclidean plane such that the distance $PA, PB, PC, PD$ are all rational numbers, where $ABCD$ is a unit square $A(0, 0), B(1, 0), C(0, 1), D(1, 1)$.

A first observation is that $P$ has to be a rational point, i.e. $x, y \in \mathbb{Q}$.

*Proof.* $1 - 2x = (1 - x)^2 - x^2 = PB^2 - PA^2$ is rational. Same for $y$. □

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For the time being, I want to interpret this problem in the language of projective varieties.

**Definition 1.** Let $X \subset \mathbb{P}^6 = \mathrm{Proj}\mathbb{Z}[x, y, r_1, r_2, r_3, r_4, z]$ be the projective variety defined by the homogeneous ideal $I(X) = < x^2+y^2-r_1^2, (x-z)^2+y^2-r_2^2, x^2+(y-z)^2-r_3^2, (x-z)^2+(y-z)^2-r_4^2 >$.

Our goal is to find rational points on $X$ where $z = 1$. For the sake of convenience, I restrict the base field to be $\mathbb{R}$ firstly, and I'll discuss about the topology of $X(\mathbb{R})$.

**Definition 2.** We define the projection map to be $f : X(\mathbb{R}) \to \mathbb{RP}^2 = \mathrm{Proj}\mathbb{R}[x, y, z], p(x : y : r_1 : r_2 : r_3 : r_4 : z) \mapsto (x : y : z)$.

Some properties of $f$ are given as follows:

**Proposition 1.** i) $f$ is well defined. Suppose $p(x:y:r_1:r_2:r_3:r_4:z) \in X$, and $t \in \mathbb{R}_\times$, then $f(p) = (x:y:z) = (tx:ty:tz) = f(tp)$. Moreover, for any $p \in X$, we have $(x_p:y_p:z_p) \neq (0:0:0)$ which does not belong to $\mathbb{P}^2$. So $f$ is well defined indeed.

ii) Consider the coordinate chart $\{z \neq 0\}$ on $\mathbb{RP}^6$ and identify this chart to $\mathbb{R}^6$, we may assume $z_p = 1$ for all such points $p \in z \neq 0 \subset \mathbb{P}^6$. Let

$$\begin{aligned} &q_1(0:0:1), \quad q_2(1:0:1), \quad q_3(0:1:1), \quad q_4(1:1:1) \\ &p_1(0:0:0:\pm1:\pm1:\pm\sqrt{2}:1), \quad p_2(1:0:\pm1:0:\pm\sqrt{2}:\pm1:1), \\ &p_3(0:1:\pm1:\pm\sqrt{2}:0:\pm1:1), \quad p_4(1:1:\pm\sqrt{2}:\pm1:\pm1:0:1) \end{aligned} \tag{1}$$

then clearly $p_i = f^{-1}(q_i)$ gives the fiber of $q_i, 1 \le i \le 4$. We claim that the set $\{f^{-1}(q_i), 1 \le i \le 4\} = \{p_i, 1 \le i \le 4\}$ of 32 points are the only points where $f$ acts singularly locally.

On the chart $\{z \neq 0\} \subset \mathbb{RP}^6$ we assume $z = 1$, let $p(x:y:r_1:r_2:r_3:r_4:1) \in X, q(x:y:1) = f(p)$ and $p \neq p_i, 1 \le i \le 4$, then $q \neq q_i, 1 \le i \le 4$. We claim that locally $f$ is a smooth diffeomorphism on a neighbourhood of $p$ in $X$. The fiber of $q$ is given by $f^{-1}(q) = \{(x:y:\pm\sqrt{x^2+y^2}:\pm\sqrt{(x-1)^2+y^2}:\pm\sqrt{x^2+(y-1)^2}:\pm\sqrt{(x-1)^2+(y-1)^2}:1)\}$.

Moreover, on the chart $\{y \neq 0\} \subset \mathbb{RP}^6$, we assume $y = 1$, let $p(x:1:r_1:r_2:r_3:r_4:z) \in X, q(x:1:z) = f(p)$ and $p \neq p_3, p_4$, then $q \neq q_3, q_4, z \neq 1$. Since $x^2+1 = r_1^2 \neq 0, \quad (x-z)^2+1 = r_2^2 \neq 0, \quad x^2+(1-z)^2 = r_3^2 \neq 0, \quad (x-z)^2+(1-z)^2 = r_4^2 \neq 0$. Similarly, on the chart $\{x \neq 0\} \subset \mathbb{RP}^6$, we assume $x = 1$ and let $p(1:y:r_1:r_2:r_3:r_4:z) \in X, q(1:y:z) = f(p)$ and $p \neq p_2, p_4$, then $q \neq q_2, q_4, z \neq 1$. So locally $f$ is a smooth diffeomorphism on a neighbourhood of $p$ in $X$ in both the above cases. Thus we see that $f$ is a 16-fold covering from $X(\mathbb{R})\backslash\{p_i, 1 \le i \le 4\}$ to $\mathbb{RP}^2\backslash\{q_i, 1 \le i \le 4\}$.

We also consider the deformation of $X(\mathbb{R})$ parametrized by a real number $\epsilon$.

**Definition 3.** Let $X_\epsilon \subset \mathbb{RP}^6 = \mathrm{Proj}\mathbb{R}[x,y,r_1,r_2,r_3,r_4,z]$ be the projective variety defined by the homogeneous ideal $I(X_\epsilon) = < x^2+y^2+\epsilon z^2 - r_1^2, (x-z)^2+y^2+\epsilon z^2 - r_2^2, x^2+(y-z)^2+\epsilon z^2 - r_3^2, (x-z)^2+(y-z)^2+\epsilon z^2 - r_4^2 >$. Note that in the definition above, the base field for $X_\epsilon$ is $\mathbb{R}$, whereas $X$ is defined over $\mathrm{Spec}\mathbb{Z}$. We first consider the real points $X_\epsilon(\mathbb{R})$ of $X_\epsilon$. But actually, $X_\epsilon$ can be regarded as a variety over $\mathbb{Q}$ if we choose $\epsilon \in \mathbb{Q}$.

Let $f_\epsilon : X_\epsilon(\mathbb{R}) \to \mathbb{RP}^2, p \mapsto (x_p:y_p:z_p)$, here $\epsilon$ can take every real number value. We claim that when $\epsilon > 0$, $f_\epsilon$ is locally a smooth diffeomorphism everywhere on $X_\epsilon(\mathbb{R})$, and globally it's a 16-fold covering from $X_\epsilon$ to $\mathbb{RP}^2$. Moreover, $X_\epsilon(\mathbb{R})$ can be decomposed into the union of 8 disjoint connected components as $X_\epsilon(\mathbb{R}) = \bigsqcup_{0 \le i \le 7} X_{\epsilon,i}$ where each $X_{\epsilon,i}$ is diffeomorphic to a 2-dimensional sphere $\mathbb{S}^2$, and the exact form of the point set $X_{\epsilon,i}$ are given in the proof.

*Proof.* i) First, we show that $f_\epsilon$ is well defined for any $\epsilon \in \mathbb{R}$. Since for any $p \in X_\epsilon(\mathbb{R}), t \in \mathbb{R}_\times, f_\epsilon(p) = f_\epsilon(tp)$, and $(x_p:y_p:z_p) \neq (0:0:0)$, it is true.

ii) When $\epsilon > 0$, we choose three charts $\{z \neq 0\}, \{y \neq 0\}, \{x \neq 0\} \subset \mathbb{RP}^6$. These three charts covers all the points of $X_\epsilon(\mathbb{R})$. We show that $f_\epsilon$ is locally a smooth diffeomorphism on each of the three charts above.

On the chart $\{z \neq 0\} \subset \mathbb{RP}^6$, let $z_p = 1$ for all the points $p$ in this chart. For any $p(x:y:r_1:r_2:r_3:r_4:1) \in X_\epsilon(\mathbb{R})$, we show that $f_\epsilon$ is locally a diffeomorphism. $f_\epsilon(p) = (x:y:1) \in \mathbb{RP}^2$, its inverse is given by $(x:y:1) \mapsto (x:y:\pm\sqrt{x^2+y^2+\epsilon}:\pm\sqrt{(x-1)^2+y^2+\epsilon}:\pm\sqrt{x^2+(y-1)^2+\epsilon}:\pm\sqrt{(x-1)^2+(y-1)^2+\epsilon}:1)$ for properly chosen signs in front of $r_1, r_2, r_3, r_4$ locally. Note that on this chart, $f_\epsilon$ is a 16-fold covering.

Similarly, on the chart $\{y \neq 0\} \subset \mathbb{RP}^6$, let $z_p = 1$ for all the points $p$ in this chart. For any $p(x : 1 : r_1 : r_2 : r_3 : r_4 : z) \in X_\epsilon(\mathbb{R})$, $f_\epsilon(p) = (x : 1 : z) \in \mathbb{RP}^2$, its inverse is given by $(x : 1 : z) \mapsto (x : 1 : \pm\sqrt{x^2 + 1 + \epsilon z^2} : \pm\sqrt{(x-z)^2 + 1 + \epsilon z^2} : \pm\sqrt{x^2 + (1-z)^2 + \epsilon z^2} : \pm\sqrt{(x-z)^2 + (1-z)^2 + \epsilon z^2} : 1)$ for properly chosen signs in front of $r_1, r_2, r_3, r_4$ locally. So $f_\epsilon$ is a local diffeomorphism.

On the chart $\{x \neq 0\} \subset \mathbb{RP}^6$, the case is similar to the case for $\{y \neq 0\}$, and we omit the proof here. Note that $f_\epsilon$ is also a 16-fold covering on both charts $\{y \neq 0\}$ and $\{x \neq 0\}$. We conclude that $f_\epsilon$ is locally a smooth diffeomorphism everywhere on $X_\epsilon(\mathbb{R})$, and globally it's a 16-fold covering from $X_\epsilon$ to $\mathbb{RP}^2$.

iii) We give the connected components decomposition of $X_\epsilon(\mathbb{R})$ as $X_\epsilon(\mathbb{R}) = \bigsqcup_{0 \leq i \leq 7} X_{\epsilon,i}$, and show that each $X_{\epsilon,i}$ is diffeomorphic to a 2-dimensional sphere $\mathbb{S}^2$. Since $\epsilon > 0$, $X_\epsilon(\mathbb{R})$ lies totally in the chart $\{r_1 \neq 0\} \subset \mathbb{RP}^6$. Define a map $g_1 : X_\epsilon(\mathbb{R}) \to \mathbb{R}^3$, for any $p(x : y : r_1 : r_2 : r_3 : r_4 : z) \in X_\epsilon(\mathbb{R}), g_1(p) = (\frac{x}{r_1}, \frac{y}{r_1}, \frac{z}{r_1})$. For any $t \in \mathbb{R}_\times, g_1(tp) = (\frac{tx}{tr_1}, \frac{ty}{tr_1}, \frac{tz}{tr_1}) = g_1(p)$, so $g_1$ is well defined. The image of $g_1$ is $g_1(X_\epsilon(\mathbb{R})) = \mathbb{S}^2_{\epsilon,1} = \{x^2 + y^2 + \epsilon z^2 = 1\} \subset \mathbb{R}^3$. For a given point $s_1(x, y, z) \in \mathbb{S}^2_{\epsilon,1}$, its fiber is given by $g_1^{-1}(s_1) = \{(x : y : 1 : \pm\sqrt{x^2 + y^2 + \epsilon z^2} : \pm\sqrt{(x-z)^2 + y^2 + \epsilon z^2} : \pm\sqrt{x^2 + (y-z)^2 + \epsilon z^2} : \pm\sqrt{(x-z)^2 + (y-z)^2 + \epsilon z^2} : z)\}$. We conclude that $g_1 : X_\epsilon(\mathbb{R}) \to \mathbb{S}^2_{\epsilon,1} \subset \mathbb{R}^3 \backslash \{0\}$ is an 8-fold covering, and its locally a smooth diffeomorphism everywhere on $X_\epsilon(\mathbb{R})$. By the fiber coordinate formula given above, we know that $X_\epsilon(\mathbb{R})$ can be decomposed into 8 connected components according to the combinations of the signs of $(\frac{r_2}{r_1}, \frac{r_3}{r_1}, \frac{r_4}{r_1})$. We may write this decomposition as $X_\epsilon(\mathbb{R}) = \bigsqcup_{0 \leq i \leq 7} X_{\epsilon,i}$, where the exact correspondence between $X_{\epsilon,i}, 0 \leq i \leq 7$ and the signs of $(\frac{r_2}{r_1}, \frac{r_3}{r_1}, \frac{r_4}{r_1})$ are given in the following table:

| i | sgn($\frac{r_2}{r_1}$) | sgn($\frac{r_3}{r_1}$) | sgn($\frac{r_4}{r_1}$) |
|---|---|---|---|
| 0 | + | + | + |
| 1 | - | + | + |
| 2 | + | - | + |
| 3 | - | - | + |
| 4 | + | + | - |
| 5 | - | + | - |
| 6 | + | - | - |
| 7 | - | - | - |

Table 1: Correspondence between $X_{\epsilon,i}$ and the signs of $(\frac{r_2}{r_1}, \frac{r_3}{r_1}, \frac{r_4}{r_1})$

iv) What's more, we can deal with $X_\epsilon(\mathbb{R})$ similarly in the charts $\{r_2 \neq 0\}, \{r_3 \neq 0\}, \{r_4 \neq 0\}$ as what we have done in the chart $\{r_1 \neq 0\}$. $X_\epsilon(\mathbb{R})$ lies totally in each of the above three charts. We may define $g_2, g_3, g_4 : X_\epsilon(\mathbb{R}) \to \mathbb{R}^3$ as follows: for any $p(x : y : r_1 : r_2 : r_3 : r_4 : z) \in X_\epsilon(\mathbb{R})$, let $g_2(p) = (\frac{x}{r_2}, \frac{y}{r_2}, \frac{z}{r_2})$, $g_3(p) = (\frac{x}{r_3}, \frac{y}{r_3}, \frac{z}{r_3})$, $g_4(p) = (\frac{x}{r_4}, \frac{y}{r_4}, \frac{z}{r_4})$. Their images are given by $\mathrm{Im}(g_2) = \mathbb{S}^2_{\epsilon,2} = \{(x-z)^2 + y^2 + \epsilon z^2 = 1\}$, $\mathrm{Im}(g_3) = \mathbb{S}^2_{\epsilon,3} = \{x^2 + (y-z)^2 + \epsilon z^2 = 1\}$, $\mathrm{Im}(g_4) = \mathbb{S}^2_{\epsilon,4} = \{(x-z)^2 + (y-z)^2 + \epsilon z^2 = 1\}$ □

A useful observation is that the topology of $X_\epsilon(\mathbb{R})$ for positive $\epsilon$ is pretty simple. In the following section, I'll show that we can deduce a CW-complex stucture of $X(\mathbb{R})$ by taking the limit $\epsilon \to 0_+$ for $\epsilon > 0$.

## 2 Topological inspection of $X(\mathbb{R})$ and $X_\epsilon(\mathbb{R})$

Recall that $f_\epsilon : X_\epsilon(\mathbb{R}) \to \mathbb{RP}^2, p \mapsto (x_p : y_p : z_p)$ is a 16-fold smooth covering. $f : X(\mathbb{R}) \to \mathbb{RP}^2, p \mapsto (x_p : y_p : z_p)$ is a 16-fold locally smooth covering almost everywhere on $X(\mathbb{R})$ except for 32 exceptional points. To be exact, these 32 exceptional points on $X(\mathbb{R})$ are given by $p_i = f^{-1}(q_i)$, and the exact coordinates of $p_i, q_i, 1 \le i \le 4$ are given by equation 1 in the first section. We assign indices to points in $f_\epsilon^{-1}(q_1)$ as follows:

**Definition 4.** Assume $p(x : y : r_1 : r_2 : r_3 : r_4 : z) = p_{1,i,j}$ to be a point in the fiber $f_\epsilon^{-1}(q_1)$ where $0 \le i \le 7, 0 \le j \le 1$. Its indices $(i,j)$ can be uniquely determined as follows: $i$ is the index of the connected component $X_{\epsilon,i}$ where $p$ lies on, which is determined by the signs of $(\frac{r_2}{r_1}, \frac{r_3}{r_1}, \frac{r_4}{r_1})$. Put $j = 0$ for $\mathrm{sgn}(\frac{z}{r_1}) = 1$, and $j = 1$ for $\mathrm{sgn}(\frac{z}{r_1}) = -1$. Note that since $f_\epsilon(p) = q_1$, we must have $z \ne 0$. So $\mathrm{sgn}(\frac{z}{r_1}) = \pm 1$ are the only possible cases. Conversely, for every pair of such indices $(i,j), 0 \le i \le 7, 0 \le j \le 1$, we can select a point $p \in f_\epsilon^{-1}(q_1)$ such that its indices determined as the process above are $(i,j)$. So we have $f_\epsilon^{-1}(q_1) = \{p_{1,i,j}, 0 \le i \le 7, 0 \le j \le 1\}$. More precisely, we have:

$$\begin{aligned}
&p_{1,0,0}(0:0:\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1), \quad p_{1,0,1}(0:0:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1),\\
&p_{1,1,0}(0:0:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1), \quad p_{1,1,1}(0:0:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1),\\
&p_{1,2,0}(0:0:\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1), \quad p_{1,2,1}(0:0:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1),\\
&p_{1,3,0}(0:0:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1), \quad p_{1,3,1}(0:0:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1),\\
&p_{1,4,0}(0:0:\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1), \quad p_{1,4,1}(0:0:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1),\\
&p_{1,5,0}(0:0:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1), \quad p_{1,5,1}(0:0:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1),\\
&p_{1,6,0}(0:0:\sqrt{\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1), \quad p_{1,6,1}(0:0:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1),\\
&p_{1,7,0}(0:0:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:1), \quad p_{1,7,1}(0:0:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:1).
\end{aligned} \tag{2}$$

We give indices to each $p_1$(this is actually a set of 8 points till now) in the following way:

$$\begin{aligned}
p_{1,0}(0:0:0:1:1:\sqrt{2}:1), &\quad p_{1,1}(0:0:0:-1:1:\sqrt{2}:1),\\
p_{1,2}(0:0:0:1:-1:\sqrt{2}:1), &\quad p_{1,3}(0:0:0:-1:-1:\sqrt{2}:1),\\
p_{1,4}(0:0:0:1:1:-\sqrt{2}:1), &\quad p_{1,5}(0:0:0:-1:1:-\sqrt{2}:1),\\
p_{1,6}(0:0:0:1:-1:-\sqrt{2}:1), &\quad p_{1,7}(0:0:0:-1:-1:-\sqrt{2}:1).
\end{aligned} \tag{3}$$

Notice that in the notations given above, the index $i$ of $p_{1,i}$ bijectively corresponds to the signs of $(r_2, r_3, r_4)$.

Note that in the notations above, $p_{1,i,j}$ is dependent on $\epsilon$. We omit $\epsilon$ in the notations above since there won't be any ambiguities for the time being. The following theorem gives a CW-complex structure on $X(\mathbb{R})$ induced by that of $X_\epsilon(\mathbb{R})$ for $\epsilon > 0$, and take the limit $\epsilon \to 0_+$.

**Proposition 2.** From the discussions given above, for any $0 \le i \le 7$, we have $\lim\limits_{\epsilon \to 0_+} p_{1,i,0} = \lim\limits_{\epsilon \to 0_+} p_{1,i \wedge 7,1} = p_{1,i}$, where $\wedge$ denotes the bitwise xor operation for binary integers.

*Proof.* It follows directly if we check the closed form of coordinates of $\{p_{1,i,j},$ and $p_{1,i}, 0 \le i \le 7, 0 \le j \le 1\}$ given above carefully. □

The following theorem shows the way of adjunctions that joins the fibers $f_\epsilon^{-1}(q_2), f_\epsilon^{-1}(q_3), f_\epsilon^{-1}(q_4)$, to the fibers $f^{-1}(q_2), f^{-1}(q_3), f^{-1}(q_4)$ respectively. It's statement is similar to the definition 4 and proposition 2 above.

**Definition 5.** 1) Assume $p(x : y : r_1 : r_2 : r_3 : r_4 : z) = p_{2,i,j}$ to be a point on the fiber $f_\epsilon^{-1}(q_2)$ where $0 \le i \le 7, 0 \le j \le 1$ are uniquely determined as in the definition 4 above, that is, $p_{2,i,j} \in X_{\epsilon,i}$ is the connected component that $p$ lies on, $j = 0$ for $\mathrm{sgn}(\frac{z}{r_2}) = 1$, and $j = 1$ for $\mathrm{sgn}(\frac{z}{r_2}) = -1$. More precisely, we have:

$$\begin{aligned}
&p_{2,0,0}(1:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{2,0,1}(1:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1),\\
&p_{2,1,0}(1:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{2,1,1}(1:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{2,2,0}(1:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{2,2,1}(1:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1),\\
&p_{2,3,0}(1:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{2,3,1}(1:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{2,4,0}(1:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{2,4,1}(1:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{2,5,0}(1:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{2,5,1}(1:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1),\\
&p_{2,6,0}(1:0:\sqrt{1+\epsilon}:\sqrt{\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{2,6,1}(1:0:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{2,7,0}(1:0:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{2,7,1}(1:0:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:1).
\end{aligned} \tag{4}$$

We give indices to each $p_2$(this is a set of 8 points till now) in the following way:

$$\begin{aligned}
&p_{2,0}(1:0:1:0:\sqrt{2}:1:1), \quad p_{2,1}(1:0:-1:0:-\sqrt{2}:-1:1),\\
&p_{2,2}(1:0:1:0:-\sqrt{2}:1:1), \quad p_{2,3}(1:0:-1:0:\sqrt{2}:-1:1),\\
&p_{2,4}(1:0:1:0:\sqrt{2}:-1:1), \quad p_{2,5}(1:0:-1:0:-\sqrt{2}:1:1),\\
&p_{2,6}(1:0:1:0:-\sqrt{2}:-1:1), \quad p_{2,7}(1:0:-1:0:\sqrt{2}:1:1).
\end{aligned} \tag{5}$$

Notice that in the notations given above, the index $i$ of $p_{1,i}$ bijectively corresponds to the signs of $(r_1, r_3, r_4)$, but the exact correspondence here is non-trivial.

2) Similarly, when $p(x : y : r_1 : r_2 : r_3 : r_4 : z) = p_{3,i,j}$ is a point on the fiber $f_\epsilon^{-1}(q_3)$ where $0 \le i \le 7, 0 \le j \le 1$, its indices $(i, j)$ are uniquely determined as: $p_{3,i,j} \in X_{\epsilon,i}$ is the connected component that $p$ lies on, $j = 0$ for $\mathrm{sgn}(\frac{z}{r_3}) = 1$, and $j = 1$ for $\mathrm{sgn}(\frac{z}{r_3}) = -1$. More precisely, we have:

$$\begin{aligned}
&p_{3,0,0}(0:1:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{3,0,1}(0:1:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1),\\
&p_{3,1,0}(0:1:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{3,1,1}(0:1:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1),\\
&p_{3,2,0}(0:1:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{3,2,1}(0:1:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{3,3,0}(0:1:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{3,3,1}(0:1:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{3,4,0}(0:1:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{3,4,1}(0:1:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{3,5,0}(0:1:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1), \quad p_{3,5,1}(0:1:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:-\sqrt{\epsilon}:\sqrt{1+\epsilon}:1),\\
&p_{3,6,0}(0:1:-\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{3,6,1}(0:1:\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1),\\
&p_{3,7,0}(0:1:-\sqrt{1+\epsilon}:\sqrt{2+\epsilon}:\sqrt{\epsilon}:\sqrt{1+\epsilon}:1), \quad p_{3,7,1}(0:1:\sqrt{1+\epsilon}:-\sqrt{2+\epsilon}:-\sqrt{\epsilon}:-\sqrt{1+\epsilon}:1).
\end{aligned} \tag{6}$$

We give indices to each $p_3$(this is a set of 8 points till now) in the following way:

$$\begin{aligned} p_{3,0}(0:1:1:\sqrt{2}:0:1:1), &\quad p_{3,1}(0:1:1:-\sqrt{2}:0:1:1),\\ p_{3,2}(0:1:-1:-\sqrt{2}:0:-1:1), &\quad p_{3,3}(0:1:-1:\sqrt{2}:0:-1:1),\\ p_{3,4}(0:1:1:\sqrt{2}:0:-1:1), &\quad p_{3,5}(0:1:1:-\sqrt{2}:0:-1:1),\\ p_{3,6}(0:1:-1:-\sqrt{2}:0:1:1), &\quad p_{3,7}(0:1:-1:\sqrt{2}:0:1:1). \end{aligned} \tag{7}$$

Notice that in the notations given above, the index $i$ of $p_{3,i}$ bijectively corresponds to the signs of $(r_1, r_2, r_4)$, but the exact correspondence here is non-trivial.

3) Similarly, when $p(x:y:r_1:r_2:r_3:r_4:z) = p_{4,i,j}$ is a point on the fiber $f_\epsilon^{-1}(q_4)$ where $0 \le i \le 7, 0 \le j \le 1$, its indices $(i,j)$ are uniquely determined as: $p_{4,i,j} \in X_{\epsilon,i}$ is the connected component that $p$ lies on, $j=0$ for $\mathrm{sgn}(\frac{z}{r_4}) = 1$, and $j=1$ for $\mathrm{sgn}(\frac{z}{r_4}) = -1$. More precisely, we have:

$$\begin{aligned} p_{4,0,0}(1:1:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,0,1}(1:1:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\ p_{4,1,0}(1:1:\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,1,1}(1:1:-\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\ p_{4,2,0}(1:1:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,2,1}(1:1:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\ p_{4,3,0}(1:1:\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,3,1}(1:1:-\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\ p_{4,4,0}(1:1:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,4,1}(1:1:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\ p_{4,5,0}(1:1:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,5,1}(1:1:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\ p_{4,6,0}(1:1:-\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,6,1}(1:1:\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1),\\ p_{4,7,0}(1:1:-\sqrt{2+\epsilon}:\sqrt{1+\epsilon}:\sqrt{1+\epsilon}:\sqrt{\epsilon}:1), &\quad p_{4,7,1}(1:1:\sqrt{2+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{1+\epsilon}:-\sqrt{\epsilon}:1). \end{aligned} \tag{8}$$

We give indices to each $p_4$(this is a set of 8 points till now) in the following way:

$$\begin{aligned} p_{4,0}(1:1:\sqrt{2}:1:1:0:1), &\quad p_{4,1}(1:1:\sqrt{2}:-1:1:0:1),\\ p_{4,2}(1:1:\sqrt{2}:1:-1:0:1), &\quad p_{4,3}(1:1:\sqrt{2}:-1:-1:0:1),\\ p_{4,4}(1:1:-\sqrt{2}:-1:-1:0:1), &\quad p_{4,5}(1:1:-\sqrt{2}:1:-1:0:1),\\ p_{4,6}(1:1:-\sqrt{2}:-1:1:0:1), &\quad p_{4,7}(1:1:-\sqrt{2}:1:1:0:1). \end{aligned} \tag{9}$$

Notice that in the notations given above, the index $i$ of $p_{4,i}$ bijectively corresponds to the signs of $(r_1, r_2, r_3)$, but the exact correspondence here is non-trivial.

Note that in the notations above, $p_{1,i,j}$ is dependent on $\epsilon$. We omit $\epsilon$ in the notations above since there won't be any ambiguities for the time being. The following theorem gives a CW-complex structure on $X(\mathbb{R})$ induced by that of $X_\epsilon(\mathbb{R})$ for $\epsilon > 0$, and take the limit $\epsilon \to 0_+$.

**Proposition 3.** From the discussions given above, for any $0 \le i \le 7$, we have:

i) $\lim\limits_{\epsilon\to 0_+} p_{2,i,0} = \lim\limits_{\epsilon\to 0_+} p_{2,i\wedge 1,1} = p_{2,i}$; ii) $\lim\limits_{\epsilon\to 0_+} p_{3,i,0} = \lim\limits_{\epsilon\to 0_+} p_{3,i\wedge 2,1} = p_{3,i}$;

iii) $\lim\limits_{\epsilon\to 0_+} p_{4,i,0} = \lim\limits_{\epsilon\to 0_+} p_{4,i\wedge 4,1} = p_{4,i}$;

where $\wedge$ denotes the bitwise xor operation for binary integers.

*Proof.* It follows directly if we check the exact form of coordinates of $p_{2,i,j}, p_{3,i,j}, p_{4,i,j}$, and $p_{2,i}, p_{3,i}, p_{4,i}$ for indices $0 \le i \le 7, 0 \le j \le 1$ given above carefully. $\square$

We are going to study the deformation $X_\epsilon(\mathbb{R})$ of $X(\mathbb{R})$ in the case when $\epsilon < 0$. Notice that actually we have the identity $X(\mathbb{R}) = X_0(\mathbb{R})$. The behavior of $f_\epsilon$ when $\epsilon < 0$ is more complicated than the cases when $\epsilon > 0$ or $\epsilon = 0$. To get some insight to this scene, we first study a similar toy model on $\mathbb{RP}^3$.

**Definition 6** (**Toy Model**)**.** We consider the deformation of cone singularity at the origin of $\mathbb{R}^3$. Let $Y \subset \mathbb{P}^3 = \mathrm{Proj}\mathbb{Z}[x, y, r_1, z]$ be the projective variety defined by the homogeneous ideal $I(Y) =< x^2 + y^2 - r_1^2 >$. Let $Y_\epsilon \subset \mathbb{RP}^3 = \mathrm{Proj}\mathbb{R}[x, y, r_1, z]$ be the projective variety defined by the homogeneous ideal $I(Y_\epsilon) =< x^2 + y^2 + \epsilon z^2 - r_1^2 >$.

**Theorem 1.** 1) In the toy model given above, i) when $\epsilon > 0$, $Y_\epsilon(\mathbb{R})$ is diffeomorphic to the 2-dimensional unit sphere in $\mathbb{R}^3$.

ii) When $\epsilon = 0$, $Y_0(\mathbb{R}) = Y(\mathbb{R})$ is diffeomorphic to a 2-dimensional unit sphere with north and south poles adjoined. We take this example as a typical model for all cone singularities.

iii) When $\epsilon < 0$, $Y_\epsilon(\mathbb{R})$ is diffeomorphic to a 2-dimensional torus, which is the product variety of two unit circles $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$.

2) Return to the case of $X_\epsilon(\mathbb{R})$ where $\epsilon < 0$, we claim that $X_\epsilon(\mathbb{R})$ is a smooth connected 2-dimensional variety diffeomorphic to a genus 25 orientable surface. Caution: *The proofs of arguments ii) and 2) are done intuitively, more rigorous proofs are needed in the future.*

*Proof.* 1) Part i) is very simple, embed $Y_\epsilon(\mathbb{R})$ inside the chart $\{r_1 \neq 0\} \subset \mathbb{RP}^3$ gives the required diffeomorphism. More precisely, it's given by $g_1 : Y_\epsilon(\mathbb{R}) \to \mathbb{R}^3$, for any $p(x : y : r_1 : z) \in Y_\epsilon(\mathbb{R}), g_1(p) = (\frac{x}{r_1}, \frac{y}{r_1}, \frac{z}{r_1})$. $g_1$ is well defined since its homogeneous of degree 0, and $r_1 \neq 0$ for any $p \in Y_\epsilon(\mathbb{R})$.

ii) We let $\epsilon$ vary continuously from some positive real value to 0, and consider the deformation of $Y_\epsilon(\mathbb{R})$. Identify $X_\epsilon(\mathbb{R})$ in the chart $\{r_1 \neq 0\}$ with the image of $g_1$ in $\mathbb{R}^3$, let $n(0, 0, \frac{1}{\sqrt{\epsilon}}), s(0, 0, -\frac{1}{\sqrt{\epsilon}}) \in X_\epsilon(\mathbb{R})$ be its north and south poles respectively. Their homogeneous coordinates are $n(0 : 0 : \sqrt{\epsilon} : 1), s(0 : 0 : -\sqrt{\epsilon} : 1) \in \mathbb{RP}^3$. Change the chart to be $\{z \neq 0\}$ and set $z = 1$, we see that the coordinates of the poles are $n(0, 0, \sqrt{\epsilon}), s(0, 0, -\sqrt{\epsilon})$ in this chart. Let $o(0 : 0 : 0 : 1) \in Y(\mathbb{R})$ be the cone singularity, When $\epsilon \to 0_+$, we have $n, s \to o$, in this process we see that .

iii) For any $p(x : y : r_1 : z) \in Y_\epsilon(\mathbb{R})$, let $r = \sqrt{x^2 + y^2} = \sqrt{r_1^2 - \epsilon z^2}$, we see that $r > 0$ holds for any $p \in Y_\epsilon(\mathbb{R})$. Define $g_1 : Y_\epsilon(\mathbb{R}) \to \mathbb{T}_\epsilon/\pm, p \mapsto (\frac{x}{r} : \frac{y}{r} : \frac{r_1}{r} : \frac{z}{r})$, where $\mathbb{T}_\epsilon = \{x^2 + y^2 = r^2 - \epsilon z^2 = 1\} \subset \mathbb{R}^4$, $\pm$ is the equivalence relation $p \sim -p$. Actally $\mathbb{T}_\epsilon/\pm$ is the image of $\mathbb{T}_\epsilon$ through the canonical quotient map $/\mathbb{R}_\times : \mathbb{R}^4 \backslash \{0\} \to \mathbb{RP}^3$. To see that $g_1$ is well defined, take $p \in Y_\epsilon(\mathbb{R}), t \in \mathbb{R}_\times$, then $g_1(tp) = (\frac{tx}{|tr|} : \frac{ty}{|tr|} : \frac{tr_1}{|tr|} : \frac{tz}{|tr|}) = \mathrm{sgn}(t)(\frac{x}{r} : \frac{y}{r} : \frac{r_1}{r} : \frac{z}{r}) = g_1(p)$. To see that $T_\epsilon/\pm$ is orientable, one argument is that the linear endomorphism $g : p \mapsto -p$ of $\mathbb{R}^4$ is orientable since its determinant is 1. Another approach is to parametrize $T_\epsilon$ by $(\mathbb{R}/2\pi\mathbb{Z})^2$. Under this parametrization, $g(\alpha, \beta) = (\alpha + \pi, \beta + \pi)$, and we can thus visualize the action of $g$ on a fundamental domain of $(\mathbb{R}/2\pi\mathbb{Z})^2$, and verify that the quotient surface is indeed orientable.

2) The above depiction of cone singularities and the topological changes that happen near these singularities during the deformation when $\epsilon \to 0$ can give us intuition on the topological structure near $p_{k,i,j}, p_{k,i}$ and $s_{k,i}$ for $1 \leq k \leq 4, 0 \leq i \leq 7, 0 \leq j \leq 1$. Proposition 2 and proposition 3 shows that locally at each of the 32 points $p_{k,i}$, we are taking connected sum while varying $\epsilon$ from positive to negative through 0. Since originally there are 8 spheres in $X_\epsilon(\mathbb{R}), \epsilon > 0$, and we know the way of doing connected sums on these spheres, we conclude that the result manifold $X_\epsilon(\mathbb{R}), \epsilon < 0$ is connected and has genus 25. *Question: Why is it orientable?* □

**Corollary 1.** i) When $\epsilon > 0$, the $\mathbb{Z}$ coefficient homology groups of $X_\epsilon(\mathbb{R})$ is given by:

$$H_2(X_\epsilon(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}^8, \quad H_1(X_\epsilon(\mathbb{R}), \mathbb{Z}) = 0, \quad H_0(X_\epsilon(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}^8.$$

ii) When $\epsilon = 0$, the $\mathbb{Z}$ coefficient homology groups of $X_0(\mathbb{R}) = X(\mathbb{R})$ is given by:

$$H_2(X(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}^8, \quad H_1(X(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}^{25}, \quad H_0(X(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}.$$

iii) When $\epsilon < 0$, the $\mathbb{Z}$ coefficient homology groups of $X_\epsilon(\mathbb{R})$ is given by:

$$H_2(X_\epsilon(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}, \quad H_1(X_\epsilon(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}^{50}, \quad H_0(X_\epsilon(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}.$$

Now we move on to study the properties of $f_\epsilon$ when $\epsilon < 0$. Recall that in proposition 1, we showed that $f_\epsilon$ is well defined for any $\epsilon \in \mathbb{R}$. But we only consider the scenerio of $\epsilon > -\frac{1}{4}$ for negative $\epsilon$ in this paper.

**Proposition 4.** 1) For $\epsilon < 0$, let $s_1, s_2, s_3, s_4 \subset \mathbb{RP}^2$ be four circles defined as follows:

$$\begin{aligned} s_1 = \{x^2 + y^2 = -\epsilon z^2\}, \quad s_2 = \{(x-z)^2 + y^2 = -\epsilon z^2\}, \\ s_3 = \{x^2 + (y-z)^2 = -\epsilon z^2\}, \quad s_4 = \{(x-z)^2 + (y-z)^2 = -\epsilon z^2\} \end{aligned} \tag{10}$$

Let $f_\epsilon^{-1}(s_k)$ be the fibers of $s_k, 1 \leq k \leq 4$. We claim that for each fixed $1 \leq k \leq 4$, $f_\epsilon^{-1}(s_k)$ can be decomposed into 8 connected components as $f_\epsilon^{-1}(s_k) = \bigsqcup_{0 \leq i \leq 7} u_{k,i}$ where for each $0 \leq i \leq 7$, $u_{k,i} \subset X_\epsilon(\mathbb{R})$ is diffeomorphic to a unit circle, and $f_\epsilon$ is an 8-fold smooth covering of $s_k$ while restricted to $f_\epsilon^{-1}(s_k)$.

2) As long as $p \in X_\epsilon(\mathbb{R})$ does not lie on any of the circles $f_\epsilon^{-1}(s_k), 1 \leq k \leq 4$, locally $f_\epsilon$ is a smooth diffeomorphism on a neighbourhood of $p$ in $X_\epsilon(\mathbb{R})$. Thus we see that globally $f_\epsilon$ is a 16-fold covering from $X(\mathbb{R}) \backslash \{u_{k,i}, 1 \leq k \leq 4, 0 \leq i \leq 7\}$ to $\mathbb{RP}^2 \backslash \bigcup_{1 \leq k \leq 4} D_k$, where $D_k = \text{int}(s_k) \cup s_k$ is closed disk domain defined by $s_k$ and its interior. $D_k$ has radius $\sqrt{-\epsilon}$ on the chart $\{z \neq 0\}$. Note that the interior of a sufficiently small circle is well defined on $\mathbb{RP}^2$.

*Proof.* 1) We show that the shape of each $s_k, 1 \leq k \leq 4$ is indeed a circle. For any fixed $1 \leq k \leq 4$, assume $p(x : y : z) \in s_k$, notice that $-\epsilon > 0$, so the equation that defines $s_k$ forces $z \neq 0$. We may set $z = 1$, then the zeros of each of the four equations form a circle on $\mathbb{R}^2$. Recall that we restrict the domain of $\epsilon$ to be $(-\frac{1}{4}, 0)$, so $s_1, s_2, s_3, s_4$ are four disjoint circles on the plane, and their interiors are disjoint. To study the shape of $f_\epsilon^{-1}(s_k)$, we take $k = 1$ as an example. For any $p(x : y : r_1 : r_2 : r_3 : r_4 : 1) \in f_\epsilon^{-1}(s_1), f_\epsilon(p) = q(x : y : 1) \in s_1 \subset \mathbb{RP}^2$. Its inverse restricted to a neighbourhood of $p$ in $f_\epsilon^{-1}(s_1)$ is given by $(x : y : 1) \mapsto (x : y : 0 : \pm\sqrt{(x-1)^2 + y^2 + \epsilon} : \pm\sqrt{x^2 + (y-1)^2 + \epsilon} : \pm\sqrt{(x-1)^2 + (y-1)^2 + \epsilon} : 1)$ for properly chosen signs of $r_2, r_3, r_4$ locally. So $f_\epsilon^{-1}(s_1)$ has 8 connected components $u_{k,i} \subset X_\epsilon(\mathbb{R}), 0 \leq i \leq 7$ depending on the signs of $r_2, r_3, r_4$ locally. Similar arguments hold for $s_2, s_3, s_4$, and we may assign the indices of $u_{k,i}$ in such a way that $u_{k,i}$ contracts to $p_{k,i} \in X(\mathbb{R})$ defined in definition 4 and definition 5 while taking the limit $\epsilon \to 0_-$, for $1 \leq k \leq 4, 0 \leq i \leq 7$.

2) Note that on the chart $\mathbb{RP}^2 \backslash \bigcup_{1 \leq k \leq 4} D_k$, $f_\epsilon$ is a 16-fold covering. We claim that the gluing functions between these 16 charts along their boundaries $u_{k,i}$ preserves their orientation, and may be viewed as taking connected sums.

□

Actually we may proceed our argument in the language of group actions. Let $G = (\mathbb{Z}/2\mathbb{Z})^4$, then $G$ acts on $X_\epsilon(\mathbb{R})$ by changing the signs of $(r_1, r_2, r_3, r_4)$.

# 3 Topological inspection of $X(\mathbb{C})$ and $X_\epsilon(\mathbb{C})$

## 3.1 Geometric invariants of complex algebraic varieties

For convenience, let $V$ be a smooth, projective and geometrically integral variety over a field $k$.

**Definition 7.** 1) Kodaira dimension $\kappa = \kappa(V) \in \{-\infty, 0, 1, ..., \dim V\}$: Let $\omega_V$ be the canonical sheaf. *Case i)* If we have $H^0(V, \omega_V^{\otimes m}) = 0$ for any integer $m \geq 1$, then define $\kappa = -\infty$. *Case ii)* If we have $H^0(V, \omega_V^{\otimes m}) \neq 0$ for some integer $m \geq 1$. Then $\kappa$ is the integer such that there exist $c_1, c_2 \in \mathbb{R}_+$ such that

$$c_1 m^\kappa \leq \dim_k H^0(V, \omega_V^{\otimes m}) \leq c_2 m^\kappa$$

holds for all $m > 0$ such that $H^0(V, \omega_V^{\otimes m}) \neq 0$.

2) Chern classes: there is a unique sequence of functions $c_1, c_2, ...$ assigning to each complex vector bundle $E \to V$ a class $c_i(E) \in H^{2i}(V, \mathbb{Z})$, depending only on the isomorphism type of $E$, such that:

a) $c_i(f^*(E)) = f^*(c_i(E))$ for a pullback bundle $f^*(E)$.

b) Via the Chern-Weil theory, a representative of each Chern class $c_k(V)$ is given as the coefficients of the characteristic polynomial of the curvature form $\Omega$ of $V$(defined by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, where $\omega$ is the connection form and $d$ is the exterior derivative):

$$\det(I + \frac{it\Omega}{2\pi}) = \sum_k c_k(V) t^k$$

3) Hodge numbers: For a compact Kähler manifold $V$ the torsion free part of the singular cohomology $H^n(V, \mathbb{Z})$ comes with a natural Hodge structure of weight $n$ given by the standard Hodge decomposition:

$$H^n(V, \mathbb{Z}) \otimes \mathbb{C} = H^n(V, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(V, \mathbb{C})$$

Here, $H^{p,q}(V, \mathbb{C})$ could either be viewed as the space of de Rham classes of degree $(p, q)$ or as the Dolbeault cohomology $H^q(V, \Omega_V^p)$

4) Suppose $E$ is a holomorphic vector bundle on $V$. For every $p \leq \dim_{\mathbb{C}} V$ we have a sheaf $\Omega^p(E)$ whose sections are holomorphic $(p, 0)$-forms with coefficients in $E$. We set

$$H^{p,q}(V, E) = H^q(V, \Omega^p(E)), \qquad h^{p,q}(V, E) = \dim_{\mathbb{C}} H^{p,q}(V, E).$$

Then the holomorphic Euler characteristics is defined by

$$\chi^p(V, E) = \sum_{q \geq 0} (-1)^q h^{p,q}(V, E)$$

**Proposition 5.** Let $V \subset \mathbb{P}_k^n$ be a nonsingular hypersurface defined by a degree-$d$ polynomial. Then the restriction map $H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}_k^n}^p) \to H^q(V, \Omega_V^p)$ is an isomorphism when $p + q < n - 1$.

*Proof.* Let $k = \mathbb{C}$, the weak Lefschetz theorem implies that the restriction map $H^i(X_{an}, \mathbb{C}) \to H^i(Y_{an}, \mathbb{C})$ is an isomorphism for $i < n - 1$. The proposition is a consequence of this together with the canonical Hodge decomposition and GAGA. □

**Theorem 2.** 1) With the help of *Macaulay2* software, we finished calculating of the Hodge diamond of $X_\epsilon(\mathbb{C})$, and it's given by:

$$h^{0,0} = 1, \quad h^{0,1} = 0, \quad h^{0,2} = 7,$$
$$h^{1,0} = 0, \quad h^{1,1} = 64, \quad h^{1,2} = 0,$$
$$h^{2,0} = 7, \quad h^{2,1} = 0, \quad h^{2,2} = 1.$$

Caution: *I'm pretty sure that these Hodge numbers from Macaulay2 aren't correct. Terry Tao's blog gave another argument to prove that $X_\epsilon(\mathbb{C})$ is a surface of general type.*

2) By examining the hodge numbers of $X_\epsilon(\mathbb{C})$ given above and the Enriques-Kodaira classification for complex algebraic surfaces, our conclusion is that, $X_\epsilon(\mathbb{C})$ is a general type surface, and it's Kodaira dimension is $\kappa(X_\epsilon(\mathbb{C})) = 2$.

*Proof.* There are 3 relations about Chern numbers and Euler characteristic of $V$ as follows:

$$c_1^2 + c_2 = 12 * \chi(V), \quad c_1^2 \leq 3c_2, \quad 5c_1^2 - c_2 + 36 \geq 0.$$

We conclude that when $V = X_\epsilon(\mathbb{C})$, $8 \leq c_1 \leq 17$. Since the Kodaira dimension $\kappa(X_\epsilon(\mathbb{C})) = 2$ when $c_1^2, c_2 > 0$, we know that $X_\epsilon(\mathbb{C})$ is a surface of general type. Remark: *Terry Tao wrote a blog about this topic, see [1]* □

### 3.2 Connections between real and complex algebraic varieties

We begin by listing some known results concerning the relationship between $V(\mathbb{R})$ and $V(\mathbb{C})$ for any projective variety $V$ over $\mathbb{R}$. Recall that the homology groups of $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are given by:

$$H_0(\mathbb{RP}^n, \mathbb{Z}) = \mathbb{Z}, \quad H_{2k-1}(\mathbb{RP}^n, \mathbb{Z}) = \mathbb{Z}_2, 1 \leq k \leq [\frac{n}{2}],$$
$$H_n(\mathbb{RP}^n, \mathbb{Z}) = \mathbb{Z} \text{ for odd } n, \quad H_n(\mathbb{RP}^n, \mathbb{Z}) = 0 \text{ for even } n,$$
$$H_{2k}(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}, 0 \leq k \leq n, \qquad H_{2k-1}(\mathbb{CP}^n, \mathbb{Z}) = 0, 1 \leq k \leq n.$$

**Theorem 3** (Thom65)**.** Let $V$ be a projective variety over $\mathbb{R}$. Then

$$\sum_i h^i(V(\mathbb{R}), \mathbb{Z}_2) \leq \sum_i h^i(V(\mathbb{C}), \mathbb{Z}_2)$$

where $h^i(V(\mathbb{R}), \mathbb{Z}_2), h^i(V(\mathbb{C}), \mathbb{Z}_2)$ denotes the dimension of the $\mathbb{Z}_2$-vector space $H^i(V(\mathbb{R}), \mathbb{Z}_2), H^i(V(\mathbb{C}), \mathbb{Z}_2)$ respectively. The equality frequently holds in this theorem. *Question: When will the equality in this theorem hold?*

Good news is that, I'm confident that my calculation for the homology groups of $X(\mathbb{R}), X_\epsilon(\mathbb{R})$ is correct. So by the above inequality, we know that the sum of the Hodge numbers of $X(\mathbb{C})$ is at least 34, and the sum of the Hodge numbers of $X_\epsilon(\mathbb{C})$ is at least 52.

**Theorem 4** (Sullivan71)**.** Let $V$ be a projective variety over $\mathbb{R}$. THen

$$\chi(V(\mathbb{R})) \equiv \chi(V(\mathbb{C})) \mod 2$$

where $\chi$ denotes the Euler characteristic. The choice of the coefficient field does not matter.

Now, all the $X_\epsilon(\mathbb{C})$ are diffeomorphic for different non-zero $\epsilon$ with sufficiently small norm.

**Proposition 6** (Warm up)**.** 1) Let $s$ be the conic curve $s = \text{Proj}\mathbb{Z}[x, y, z]/ < x^2 + y^2 - z^2 > \subset \mathbb{P}^2$, and we mainly consider its complex points $s(\mathbb{C})$. Define a map $f : s(\mathbb{C}) \to \mathbb{CP}^1, p(x : y : z) \mapsto q(x : y)$. Then $f$ is well-defined since $f(tp) = f(p)$ and $f(p) \neq (0 : 0)$. $f$ is a 2-fold covering when $q \neq q_0(-i : 1), q_1(i : 1)$. $f$ is a 1-fold covering on $q_0, q_1$ and $s(\mathbb{C}) \cap \{z = 0\} = \{q_0, q_1\}$. A CW-complex and chain complex structure of $s(\mathbb{C})$ is given by:

$$e_0^2 = \{z \neq 0, -\frac{\pi}{2} < \arg\frac{z}{x - iy} < \frac{\pi}{2}\}, \quad e_0^1 = \{z \neq 0, \arg\frac{z}{x - iy} = \frac{\pi}{2}\},$$
$$e_1^2 = \{z \neq 0, \frac{\pi}{2} < \arg\frac{z}{x - iy} < \frac{3\pi}{2}\}, \quad e_1^1 = \{z \neq 0, \arg\frac{z}{x - iy} = -\frac{\pi}{2}\},$$
$$e_0^0(-i : 1 : 0), \quad e_1^0(i : 1 : 0), \quad \partial_2 e_0^2 = e_0^1 - e_1^1, \quad \partial_2 e_1^2 = e_1^1 - e_0^1, \quad \partial_1 e_0^1 = \partial_1 e_1^1 = e_1^0 - e_0^0,$$
$$A_2 = \mathbb{Z}e_0^2 \oplus \mathbb{Z}e_1^2, \quad A_1 = \mathbb{Z}e_0^1 \oplus \mathbb{Z}e_1^1, \quad A_0 = \mathbb{Z}e_0^0 \oplus \mathbb{Z}e_1^0,$$

So its homology group is given as follows (we provide those of $s(\mathbb{R})$, which is just a unit circle, as a comparison): *Question: How to calculate its Hodge numbers?*

$$H_2(s(\mathbb{C}),\mathbb{Z})=\mathbb{Z},\quad H_1(s(\mathbb{C}),\mathbb{Z})=0,\quad H_0(s(\mathbb{C}),\mathbb{Z})=\mathbb{Z},$$
$$H_1(s(\mathbb{R}),\mathbb{Z})=\mathbb{Z},\quad H_0(s(\mathbb{R}),\mathbb{Z})=\mathbb{Z}$$

2) We also consider the degenerated conic curve $s=\mathrm{Proj}\mathbb{Z}[x,y,z]/<x^2+y^2>\subset\mathbb{P}^2$. The complex variety $s(\mathbb{C})$ is isomorphic to two complex projective lines gluing at one point. As a comparison, $s(\mathbb{R})$ is isomorphic to a single point, while for $s'=\mathrm{Proj}\mathbb{Z}[x,y,z]/<x^2-y^2>\subset\mathbb{P}^2$, $s'(\mathbb{R})$ is isomorphic to two circles gluing at one point.

$$H_2(s(\mathbb{C}),\mathbb{Z})=\mathbb{Z}^2,\quad H_1(s(\mathbb{C}),\mathbb{Z})=0,\quad H_0(s(\mathbb{C}),\mathbb{Z})=\mathbb{Z},$$
$$H_1(s'(\mathbb{R}),\mathbb{Z})=\mathbb{Z}^2,\quad H_0(s'(\mathbb{R}),\mathbb{Z})=\mathbb{Z}$$

We consider the deformation of cone singularity at the origin of $\mathbb{C}^3$. Let $Y\subset\mathbb{P}^3=\mathrm{Proj}\mathbb{Z}[x,y,r_1,z]$ be the projective variety defined by the homogeneous ideal $I(Y)=<x^2+y^2-r_1^2>$. Let $Y_\epsilon\subset\mathbb{CP}^3=\mathrm{Proj}\mathbb{C}[x,y,r_1,z]$ be the projective variety defined by the homogeneous ideal $I(Y_\epsilon)=<x^2+y^2+\epsilon z^2-r_1^2>$, here we may take $\epsilon$ to be any complex number with sufficiently small norm.

**Theorem 5.** 1) In the toy model given above, i) A CW-complex structure of $Y(\mathbb{C})$ can be given as follows: let $g:Y(\mathbb{C})\to\mathbb{CP}^2, p(x:y:r_1:z)\mapsto q(x:y:z)$. Since $g(tp)=g(p)$ and $g(p)\neq(0:0:0)$, $g$ is well-defined. Its singularities are those points in $Y(\mathbb{C})$ with $r_1=0$. Let $s=\{x^2+y^2=0\}\subset\mathbb{CP}^2$, on the chart $\{z\neq0\}$, it is the union of two lines $l_0=\{x+iy=0\}$ and $l_1=\{x-iy=0\}$. When $z=0$, $s$ includes two points at infinity of $l_0,l_1$, they are $l_{0,\infty}(-i:1:0)$ and $l_{1,\infty}(i:1:0)$. The intersection of $l_0$ and $l_1$ is $q_1(0:0:1)$. Claim: $s$ is locally smooth everywhere besides $q_1$. In the chart $\{y\neq0\}$, set $y=1$, then $s=\{p(x:1:z),x^2+1=0\}\subset\{y\neq0\}\subset\mathbb{CP}^2$ is the union of two parallel lines. Combining the depiction of $s$ in the chart $\{z\neq0\}$, our claim has been verified. So the CW-complex structure and homology of $s$ can be given as:

$$q_1(0:0:1),\quad l_0=\{x+iy=0\}\backslash\{q_1\},\quad l_1=\{x-iy=0\}\backslash\{q_1\},$$
$$A_2=\mathbb{Z}l_0\oplus\mathbb{Z}l_1,\quad A_1=0,\quad A_0=\mathbb{Z}q_1,$$
$$H_2(s,\mathbb{Z})=\mathbb{Z}^2,\quad H_1(s,\mathbb{Z})=0,\quad H_0(s,\mathbb{Z})=\mathbb{Z}.$$

$l_0,l_1$ given above contains their points at infinity, and are diffeomorphic to the complex line $\mathbb{C}^1$. All the boundary maps in the chain complex above are zero.

$g$ is a 2-fold covering from $Y(\mathbb{C})\backslash g^{-1}(s)$ to $\mathbb{CP}^2\backslash s$. For any $q(x:y:z)\in\mathbb{CP}^2\backslash s$, $g^{-1}(q)=(x:y:\sqrt{x^2+y^2}:z)$. $g$ is a 1-fold covering from $g^{-1}(s)$ to $s$. Let $\pi:\mathbb{CP}^2\backslash s\to\mathbb{CP}^1\backslash\{0,\infty\}=\mathbb{C}\backslash\{0\}$ be a fiber bundle defined by $\pi:(x:y:z)\mapsto(u:v), u=x+iy, v=x-iy$. Then $\pi$ is well-defined since $\pi(tp)=\pi(p), t\in\mathbb{C}^\times$ and $u,v\neq0$ on $\mathbb{CP}^2\backslash s$. Since $x=\frac{1}{2}(u+v), y=\frac{1}{2i}(u-v)$, its inverse is defined by $\pi^{-1}(w)=(\frac{1}{2}(u+v):\frac{1}{2i}(u-v):z)\simeq\mathbb{C}^1$ for $w(u:v)\in\mathbb{CP}^1\backslash\{0,\infty\}$. So every fiber of a single point is isomorphic to $\mathbb{C}^1$. Actually we can extend the definition of $\pi$ to $\pi:\mathbb{CP}^2\backslash\{q_1\}\to\mathbb{CP}^1,(x:y:z)\mapsto(x+iy:x-iy)$. It is well-defined outside $\{q_1\}$, and $\pi(l_0)=(0:1)=w_0,\pi(l_1)=(1:0)=w_\infty$. A CW-complex and chain complex structure of $\mathbb{CP}^1$ is given as follows:

$$w_0(0:1),\quad w_\infty(1:0),$$
$$e^1=\{w,\Re(w)<0,\Im(w)=0\},\quad e^2=\mathbb{CP}^1\backslash\{w_0,w_\infty\}\backslash e^1,$$
$$\partial_2e^2=0,\quad \partial_1e^1=w_\infty-w_0.$$

We get a CW-complex and chain complex structure of $\mathbb{CP}^2\backslash\{q_1\}$ through a lift-up along $\pi^{-1}$:

$$\begin{aligned}
\mathbb{CP}^2\backslash\{q_1\} &= e^4 \sqcup e^3 \sqcup l_0 \sqcup l_1, \\
e^4 = \pi^{-1}(e^2) &= \{(x:y:z), x+iy \neq 0, x-iy \neq 0, \arg\frac{x+iy}{x-iy} \neq \pi\}, \\
e^3 = \pi^{-1}(e^1) &= \{(x:y:z), x+iy \neq 0, x-iy \neq 0, \arg\frac{x+iy}{x-iy} = \pi\}, \\
l_0 = \pi^{-1}(w_0), \quad l_1 = \pi^{-1}(w_\infty), &\quad \partial_4 e^4 = 0, \quad \partial_3 e^3 = l_1 - l_0, \quad \partial_2 l_0 = \partial_2 l_1 = 0.
\end{aligned}$$

Moreover, we get a CW-complex and chain complex structure of $Y(\mathbb{C})$ from that of $\mathbb{CP}^2$ through a lift-up along $g^{-1}$ (remember the action of $g^{-1}$ on $s$ is $(x:y:z) \mapsto (x:y:0:z)$):

$$\begin{aligned}
Y(\mathbb{C})\backslash g^{-1}(s) = e_0^4 \sqcup e_0^3 \sqcup e_1^4 \sqcup e_1^3, &\quad g^{-1}(s) = e_0^2 \sqcup e_1^2 \sqcup e^0, \\
e_0^4 \sqcup e_1^4 &= \{(x:y:r_1:z), x+iy \neq 0, x-iy \neq 0, \arg\frac{x+iy}{x-iy} \neq \pi\}, \\
e_0^3 \sqcup e_1^3 &= \{(x:y:r_1:z), x+iy \neq 0, x-iy \neq 0, \arg\frac{x+iy}{x-iy} = \pi\}, \\
e_0^2 = g^{-1}(l_0), \quad e_1^2 &= g^{-1}(l_1), \quad e^0(0:0:0:1) = g^{-1}(q_1), \\
A_4 = \mathbb{Z}e_0^4 \oplus \mathbb{Z}e_1^4, \quad A_3 = \mathbb{Z}e_0^3 \oplus \mathbb{Z}e_1^3, &\quad A_2 = \mathbb{Z}e_0^2 \oplus \mathbb{Z}e_1^2, \quad A_1 = 0, \quad A_0 = \mathbb{Z}e^0,
\end{aligned}$$

For any $p(x:y:r_1:z) \in e_0^3 \sqcup e_1^3$, we have $x+iy \neq 0, x-iy \neq 0$, and there exist $\lambda < 0$, such that $x+iy = \lambda(x-iy)$. So $r_1^2 = (x+iy)(x-iy) = \lambda(x-iy)^2, r_1 = \pm\sqrt{\lambda}(x-iy)$. We may define $e_0^3, e_1^3$ and $e_0^4, e_1^4$ according to the argument of $\frac{r_1}{x-iy}$ (both $r_1$ and $x-iy$ are non-zero in these cells):

$$\begin{aligned}
e_0^3 &= \{(x:y:r_1:z), x+iy \neq 0, x-iy \neq 0, \arg\frac{r_1}{x-iy} = \frac{\pi}{2}\}, \\
e_1^3 &= \{(x:y:r_1:z), x+iy \neq 0, x-iy \neq 0, \arg\frac{r_1}{x-iy} = -\frac{\pi}{2}\}, \\
e_0^4 &= \{(x:y:r_1:z), x+iy \neq 0, x-iy \neq 0, -\frac{\pi}{2} < \arg\frac{r_1}{x-iy} < \frac{\pi}{2}\}, \\
e_1^4 &= \{(x:y:r_1:z), x+iy \neq 0, x-iy \neq 0, \frac{\pi}{2} < \arg\frac{r_1}{x-iy} < \frac{3\pi}{2}\}, \\
\partial_4 e_0^4 = e_1^3 - e_0^3, &\quad \partial_4 e_1^4 = e_0^3 - e_1^3, \quad \partial_3 e_0^3 = \partial_3 e_1^3 = e_1^2 - e_0^2, \quad \partial_2 e_0^2 = \partial_2 e_1^2 = 0.
\end{aligned}$$

So the homology groups of $Y(\mathbb{C})$ are

$$\begin{aligned}
H_4(Y(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}, \quad H_3(Y(\mathbb{C}),\mathbb{Z}) = 0, \quad H_2(Y(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}, \\
H_1(Y(\mathbb{C}),\mathbb{Z}) = 0, \quad H_0(Y(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}.
\end{aligned}$$

There is another route to construct the CW-complex and chain complex structure for $Y(\mathbb{C})$. Let $p_{\infty,0}(-i:1:0:0), p_{\infty,1}(i:1:0:0)$ be points in $Y(\mathbb{C})$ such that $r_1 = z = 0$. Define $\pi_1 : Y(\mathbb{C})\backslash\{p_{\infty,0}, p_{\infty,1}\} \to \mathbb{CP}^1, (x:y:r_1:z) \mapsto (r_1:z)$, then this map is well-defined. $Y(\mathbb{C})\backslash g^{-1}(s)\backslash\{z=0\}$ can be divided into 4 regions according to the argument of $\frac{r_1}{z}$ (note that we can't directly consider $\arg(r_1)$ since it's not well-defined):

$$d_0^4 = \{-\frac{\pi}{2} < \arg\frac{r_1}{z} < \frac{\pi}{2}\} = \{\Re(\frac{r_1}{z}) > 0\}, \quad d_0^3 = \{\arg\frac{r_1}{z} = \frac{\pi}{2}\} = \{\Re(\frac{r_1}{z}) = 0, \Im(\frac{r_1}{z}) > 0\},$$
$$d_1^4 = \{\frac{\pi}{2} < \arg\frac{r_1}{z} < \frac{3\pi}{2}\} = \{\Re(\frac{r_1}{z}) < 0\}, \quad d_1^3 = \{\arg\frac{r_1}{z} = -\frac{\pi}{2}\} = \{\Re(\frac{r_1}{z}) = 0, \Im(\frac{r_1}{z}) < 0\},$$
$$Y(\mathbb{C}) \cap \{r_1 = 0\} = g^{-1}(s) = d_0^2, \quad Y(\mathbb{C}) \cap \{z = 0\} = d_1^2,$$
$$d_0^2 = g^{-1}(s) = \{(x : y : 0 : z), x^2 + y^2 = 0\}, \quad d_1^2 = \{(x : y : r_1 : 0), x^2 + y^2 = r_1^2\},$$
$$d_{0,0}^2 = \{(x : y : 0 : z), x + iy = 0\}, \quad d_{0,1}^2 = \{(x : y : 0 : z), x - iy = 0\},$$
$$p_{\infty,0}(-i : 1 : 0 : 0), \quad p_{\infty,1}(i : 1 : 0 : 0), \quad Y(\mathbb{C})\backslash g^{-1}(s)\backslash\{z = 0\} = d_0^4 \sqcup d_1^4 \sqcup d_0^3 \sqcup d_1^3,$$
$$\partial_4 d_0^4 = d_0^3 - d_1^3, \quad \partial_4 d_1^4 = d_1^3 - d_0^3, \quad \partial_3 d_0^3 = \partial_3 d_1^3 = d_1^2 - d_0^2,$$

ii) When $\epsilon \neq 0$, $Y_\epsilon(\mathbb{C})$ is diffeomorphic to a complex 2-dimensional sphere. We may only consider the case when $\epsilon = -1$ since the complex structure of $Y_\epsilon(\mathbb{C})$ are all the same for different $\epsilon \neq 0$. Then the polynomial that defines $Y_\epsilon(\mathbb{C})$ becomes $x^2 + y^2 = r_1^2 + z^2$. We may rewrite it as $(x + iy)(x - iy) = (r_1 + iz)(r_1 - iz)$. By a change of variable, we see that $Y_\epsilon(\mathbb{C})$ is isomorphic to $\mathrm{Proj}\mathbb{C}[w_0, w_1, w_2, w_3]/ < w_0w_1 - w_2w_3 >$, where $w_0 = x+iy, w_1 = x-iy, w_2 = r_1+iz, w_3 = r_1-iz$. In the chart $\{w_3 \neq 0\}$, set $w_3 = 1$, then the projection $(w_0 : w_1 : w_2 : 1) \mapsto (w_0, w_1)$ is an isomorphism. Points of $Y_\epsilon(\mathbb{C})$ where $w_3 = 0$ is isomorphic to $\{w_0w_1 = 0\} \subset \mathbb{CP}^2 = \mathrm{Proj}\mathbb{C}[w_0, w_1, w_2]$, which is the union of two affine complex lines and one intersection point. The CW-complex and chain complex structure of $Y_\epsilon(\mathbb{C})$, as well as its homology groups are given by:

$$A_4 = \mathbb{Z}e^4, \quad A_3 = 0, \quad A_2 = \mathbb{Z}e_0^2 \oplus \mathbb{Z}e_1^2, \quad A_1 = 0, \quad A_0 = \mathbb{Z}e^0,$$
$$\partial_4 = \partial_3 = \partial_2 = \partial_1 = 0,$$
$$H_4(Y_\epsilon(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}, \quad H_3(Y_\epsilon(\mathbb{C}),\mathbb{Z}) = 0, \quad H_2(Y_\epsilon(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^2,$$
$$H_1(Y_\epsilon(\mathbb{C}), \mathbb{Z}) = 0, \quad H_0(Y_\epsilon(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}.$$

2) Return to the case of $X(\mathbb{C})$ and $X_\epsilon(\mathbb{C})$, their CW-complex structures are hard to compute. Define $f : X(\mathbb{C}) \to \mathbb{CP}^2, p(x : y : r_1 : r_2 : r_3 : r_4 : z) \mapsto (x : y : z) \in \mathbb{CP}^2$, then $f$ is well-defined since $f(tp) = f(p), t \in \mathbb{C}^\times$ and $f(p) \neq (0 : 0 : 0)$. For any $q(x : y : z) \in \mathbb{CP}^2$, $f^{-1}(q) = (x : y : \sqrt{x^2 + y^2} : \sqrt{(x - z)^2 + y^2} : \sqrt{x^2 + (y - z)^2} : \sqrt{(x - z)^2 + (y - z)^2} : z)$ where the square root functions above are multi-valued. Let analyse the size of the fiber $f^{-1}(q)$ for different $q \in \mathbb{CP}^2$. Denote four conic curves and their irreducible components in $\mathbb{CP}^2$ by

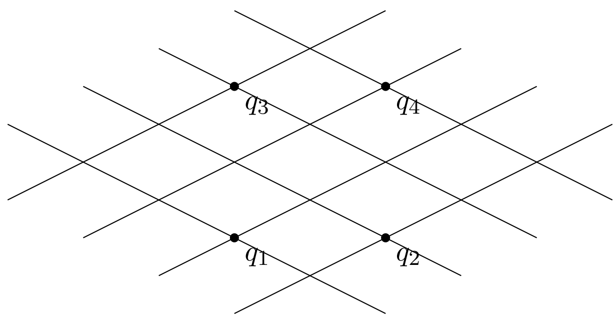
$$s_1 : x^2 + y^2 = 0, \quad l_{1,0} : x + iy = 0, \quad l_{1,1} : x - iy = 0,$$
$$s_2 : (x - z)^2 + y^2 = 0, \quad l_{2,0} : (x - z) + iy = 0, \quad l_{2,1} : (x - z) - iy = 0,$$
$$s_3 : x^2 + (y - z)^2 = 0, \quad l_{3,0} : x + i(y - z) = 0, \quad l_{3,1} : x - i(y - z) = 0,$$
$$s_4 : (x - z)^2 + (y - z)^2 = 0, \quad l_{4,0} : (x - z) + i(y - z) = 0, \quad l_{4,1} : (x - z) - i(y - z) = 0.$$

The 8 projective complex lines above has 18 joints in total, 2 of them meets 4 lines respectively, 16 of them meets 2 lines, among which 4 of them belong to single conic curve listed above. The coordinates of these points are given as follows:

$$q_{\infty,0}(-i:1:0),\quad q_{\infty,1}(i:1:0),$$
$$q_{\mathrm{db},1,2}(\frac{1}{2}:\frac{i}{2}:1),\quad q_{\mathrm{db},1,3}(-\frac{i}{2}:\frac{1}{2}:1),\quad q_{\mathrm{db},1,4}(\frac{1-i}{2}:\frac{1+i}{2}:1),$$
$$q_{\mathrm{db},2,1}(\frac{1}{2}:-\frac{i}{2}:1),\quad q_{\mathrm{db},2,3}(\frac{1-i}{2}:\frac{1-i}{2}:1),\quad q_{\mathrm{db},2,4}(\frac{2-i}{2}:\frac{1}{2}:1),$$
$$q_{\mathrm{db},3,1}(\frac{i}{2}:\frac{1}{2}:1),\quad q_{\mathrm{db},3,2}(\frac{1+i}{2}:\frac{1+i}{2}:1),\quad q_{\mathrm{db},3,4}(\frac{1}{2}:\frac{2+i}{2}:1),$$
$$q_{\mathrm{db},4,1}(\frac{1+i}{2}:\frac{1-i}{2}:1),\quad q_{\mathrm{db},4,2}(\frac{2+i}{2}:\frac{1}{2}:1),\quad q_{\mathrm{db},4,3}(\frac{1}{2}:\frac{2-i}{2}:1),$$
$$q_1(0:0:1),\quad q_2(1:0:1),\quad q_3(0:1:1),\quad q_4(1:1:1).$$

We follow the notations of $l_{i,j}, 1\le i\le 4, 0\le j\le 1$ given above, but exclude 5 intersection points from each one of them. Let $s=\bigcup_{1\le i\le 4} s_i$ be the union of the four conics above. Then a disjoint union of $s$ can be written as

$$s=\bigsqcup_{1\le i\le 4,0\le j\le 1} l_{i,j}\bigsqcup_{1\le i\le 4} q_i \bigsqcup_{1\le i\ne j\le 4} q_{\mathrm{db},i,j}\bigsqcup_{0\le j\le 1} q_{\infty,j}$$




We want to analyse the topology of $\mathbb{CP}^2\backslash s$ and $s$ in detail. Define $\pi_0:\mathbb{CP}^2\backslash\{q_{\infty,0}\}\to\mathbb{CP}^1,(x:y:z)\mapsto(x+iy:z)$, $\pi_1:\mathbb{CP}^2\backslash\{q_{\infty,1}\}\to\mathbb{CP}^1,(x:y:z)\mapsto(x-iy:z)$, then $\pi_0,\pi_1$ are well-defined since their images can't be $(0:0)$. While its domain is restricted to $\mathbb{CP}^2\backslash s$, the fiber $\pi_0^{-1}(q)$ of $\pi_0$ at any point $q(u:1)\in\mathbb{CP}^1\backslash\{\infty\}$ is a complex projective line minus 4 points. For $\infty(1:0)\in\mathbb{CP}^1$, its fiber $\pi_0^{-1}(\infty)$ is a complex projective line minus 2 points.

$f$ is 16-fold on $\mathbb{CP}^2\backslash s$, 8-fold on $s$ besides the 14 joints between different conics, 4-fold on 12 joints which each meets 2 lines, and 1-fold on 2 joints which each meets 4 lines. So formally we have the following partition of $X(\mathbb{C})$ ignoring boundary maps, and its euler characteristic is known:

$$X(\mathbb{C})=\bigsqcup 16\times(\mathbb{CP}^2\backslash s)\bigsqcup 8\times(s\backslash\{q_{\mathrm{db},*},q_{\infty,*}\})\bigsqcup 4\times q_{\mathrm{db},*}\bigsqcup q_{\infty,*},$$
$$\chi(\mathbb{CP}^2\backslash s)=9,\quad \chi(s\backslash\{q_{\mathrm{db},*},q_{\infty,*}\})=\chi(s)-\chi(q_{\mathrm{db},*})-\chi(q_{\infty,*})=-20,$$
$$\chi(q_{\mathrm{db},*})=12,\quad \chi(q_{\infty,*})=2,\quad \chi(X(\mathbb{C}))=16*9+8*(-20)+4*12+2=34.$$

$X(\mathbb{C})\backslash f^{-1}(s)\backslash\{z=0\}$ can be divided into 256 regions according to the arguments of $\frac{r_1}{z},\frac{r_2}{z},\frac{r_3}{z},\frac{r_4}{z}$. More precisely, we set $z=1$ in the chart $\{z\ne 0\}$, and divide $\mathrm{ran}(r_i)=\mathbb{C}\backslash\{0\}$ into 4 parts:

$$d_0=\{-\frac{\pi}{2}<\arg(r_i)<\frac{\pi}{2}\}=\{\Re(r_i)>0\},\quad d_1=\{\arg(r_i)=\frac{\pi}{2}\}=\{\Re(r_i)=0,\Im(r_i)>0\},$$
$$d_2=\{\frac{\pi}{2}<\arg(r_i)<\frac{3\pi}{2}\}=\{\Re(r_i)<0\},\quad d_3=\{\arg(r_i)=-\frac{\pi}{2}\}=\{\Re(r_i)=0,\Im(r_i)<0\}.$$

*Proof.* □

# 4 Rational points on $X$ and $X_\epsilon$

## 4.1 Prerequsites - Galois cohomology, class field theory and $p$-adic analysis

**Definition 8.** A number field is a finite extension of $\mathbb{Q}$. A global function field is a finite extension of $\mathbb{F}_p(t)$ for some prime $p$ or, equivalently, is the function of a geometrically integral curve over a finite field $\mathbb{F}_q$, where $q$ is a power of some prime $p$. Equivalently, a global field is the fraction field of a finitely generated $\mathbb{Z}$-algebra that is an integral domain of Krull dimension 1.

By a place of $k$, we always mean a nontrivial place of $k$. Let $\Omega_k$ be the set of places of $k$.

If $S$ is a finite nonempty subset of $\Omega_k$ containing all the archimedean places, then the ring of $S$-integers in $k$ is

$$\mathcal{O}_{k,S} = \{a \in k, v(a) \geq 0 \text{ for all } v \notin S\}$$

The adèle ring of $k$ is defined as the restricted product

$$A = A_k = \prod_{v\in\Omega_k}{}' (k_v, \mathcal{O}_v)$$

it is a $k$-alglebra for the diagonal embedding of $k$, and it is equipped with the unique topology such that: $A$ is a topological group under addition; 2) the subset $\prod_{v\in\Omega_k} \mathcal{O}_v$ is open; and 3) the subspace topology on $\prod_{v\in\Omega_k} \mathcal{O}_v$ agrees with the product topology.

The image of $k$ in $A$ is discrete, and $A/k$ is compact.

**Definition 9** (Galois cohomology)**.** If $A$ is a commutative group scheme over a field $k$, then the notion $H^q(k, A)$ denotes the Galois cohomology group $H^q(\mathrm{Gal}(k_s/k), A(k_s))$, where $k_s$ denotes the separable closure of $k$. This definition is made so as to agree with the étale cohomology group $H^q_{et}(\mathrm{Spec}k, A)$ of the sheaf defined by $A$ on the étale site of $\mathrm{Spec}k$.

*Intuitive guess:* In our example, $k = \mathbb{Q}$ and $A = (\mathbb{Z}/2\mathbb{Z})^4 = \mathrm{Spec}\mathbb{Q}[a_1, a_2, a_3, a_4]/ < a_1^2 - 1, a_2^2 - 1, a_3^2 - 1, a_4^2 - 1 >$, and the group multiplication is given by element-wise multiplication. For a group $G$ and an abelian $G$-module $M$, we may define its cohomology groups $H^q(G, M), q \geq 0$. If $G$ acts on $M$ trivially, then $H^1(G, M) = \mathrm{Hom}(G, M)$. I guess this is the case we are facing.

Note that $A = (\mathbb{Z}/2\mathbb{Z})^4$ is an abelian group, so we have $\mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}_s/\mathbb{Q}), A) = \mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}_{ab}/\mathbb{Q}), A)$. Actually we can go one more step, define the field $\mathbb{Q}^{dist} = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{13}, \sqrt{17}...)$, ($\mathbb{Q}$ with all $\sqrt{p}$ adjoined where $p = 2$ or $p \equiv 1 \mod 4$).It is the smallest field that contains all the square roots of $x^2 + y^2$, where $x, y \in \mathbb{Q}$. Here I want to quote some famous results in class field theory that might be used in this paper.

**Theorem 6** (Local and global class field theory)**.** 1) For any prime $p$, $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p) = \mathbb{Z}_p^\times$, the right hand side is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$ when $p$ is odd. When $p = 2$, $\mathbb{Z}_2^\times$ is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^\times \times \mathbb{Z}_2$.

2) Globally, we have

$$\mathrm{Gal}(\mathbb{Q}_{ab}/\mathbb{Q}) = \hat{\mathbb{Z}}^\times = \prod_{p \text{ prime}} \mathbb{Z}_p^\times$$

3) If $p_i, 1 \leq i \leq n$ are different primes that are 2 or have residue 1 modulo 4. Let $L = \mathbb{Q}(\sqrt{p_1}, ...\sqrt{p_n})$, then $\mathrm{Gal}(L/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^n$. As a result, we have

$$\mathrm{Gal}(\mathbb{Q}^{dist}/\mathbb{Q}) = \varprojlim (\mathbb{Z}/2\mathbb{Z})^n = \prod_{\text{prime } p=2 \text{ or } p\equiv 1 \mod 4} \mathbb{Z}/2\mathbb{Z}$$

*Question: does the right hand side hold? I want to give it a factorization as in 2)*

Let $G = \mathrm{Gal}(\mathbb{Q}^{dist}/\mathbb{Q})$, then for any $q \in \mathbb{QP}^2$, there is an $G$-action on the fiber $X_q = f^{-1}(q)$. But in our scenerio $A = (\mathbb{Z}/2\mathbb{Z})^4$ acts on $X_q$ by changing the signs of $r_1, r_2, r_3, r_4$, and this action includes all the possible $G$-actions on the fiber $X_q$, so the $G$-action on $X_q$ gives a homomorphism in $\mathrm{Hom}(G, A) = H^1(G, A)$. Note that usually we can only get a homomorphism from $G$ to the symmetry group of $\#X_q$ elements. The claim above restricts the image of such a homomorphism inside $A$.

**Proposition 7.** We ask the following questions on every $p$-adic local ring $\mathbb{Z}_p$: What numbers can have square roots inside $\mathbb{Z}_p$? What are the domain and range of exp, log, and $(1+x)^\alpha$ for fixed $\alpha$? Here we define the exponential and logarithm function by

$$\exp(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}, \quad \log(1+\alpha) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\alpha^n}{n}, \quad (1+x)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} x^k$$

We require that $\alpha \in \mathbb{Z}_p$. The answers are given below:
1) For $p = 2$, $r = 2^a b, v_2(b) = 0$ has a square root in $\mathbb{Z}_2$ if and only if $2|a, b \equiv 1 \mod 8$. $\exp(\alpha)$ is absolutely convergent when $v_2(\alpha) \geq 2$, $\log(1+\alpha)$ is absolutely convergent when $v_2(\alpha) \geq 1$.
2) For $p \neq 2$, $r = p^a b, v_p(b) = 0$ has a square root in $\mathbb{Z}_p$ if and only if $2|a, (\frac{b}{p}) = 1$. $\exp(\alpha)$ is absolutely convergent when $v_p(\alpha) \geq 1$, $\log(1+\alpha)$ is absolutely convergent when $v_2(\alpha) \geq 1$.
3) If $\alpha$ is a non-negative integer, then certainly $(1+x)^\alpha$ is well defined for any $x \in \mathbb{Z}_p$. For any other values of $\alpha$, $(1+x)^\alpha$ converges absolutely when $v_p(x) \geq 1$.

*Proof.* If $\alpha \in \mathbb{Z}_p$, then the binomial coefficient $\binom{\alpha}{k} \in \mathbb{Z}_p$ for all non-negative integer $k$, so the generalized binomial formula converges absolutely for $v_p(x) \geq 1$. Note that $\frac{1}{2} \in \mathbb{Z}_p$ when $p \neq 2$, so we get the analytical expansion of square root function in this case. □

## 4.2 Results assuming Bombieri-Lang conjecture

**Conjecture 1** (Bombieri-Lang)**.** Let $X$ be a smooth projective irreducible algebraic surface defined over the rationals $\mathbb{Q}$ which is of general type. Then the set $X(\mathbb{Q})$ of rational points of $X$ is not Zariski dense in $X$.

As [1] says, the Bombieri-Lang conjecture has been made for varieties of arbitrary dimension, and for more general number fields than the rationals, but the above special case of the conjecture is the only one needed for this application. The Bombieri-Lang conjecture is considered to be extremely difficult, in particular being substantially harder than Faltings' theorem, which is itself a highly non-trivial result. So this implication should not be viewed as a practical route to resolving the Erdös-Ulam problem unconditionally; rather, it is a demonstration of the power of the Bombieri-Lang conjecture.

## 4.3 Descent obstruction

**Theorem 7** (The descent obstruction to the local-global principle)**.** Let $k$ be a global field, $X$ be a $k$-variety. All the cohomologies below are fppf cohomologies. One can show that there is an injection $X(A) \to \prod_\mu X(k_\mu)$, so an element of $X(A)$ will be written as a sequence $(x_\mu)$ indexed by the places $\mu$ of $k$. The set $X(k)$ embeds diagonally into $X(A)$.

A torsor $f : Z \to X$ under a smooth affine algebraic group $G$ over $k$ restricts the locations in

$X(A)$ where rational points can lie. Namely, the commutativity of the following diagram:

$$\begin{array}{ccc} X(k) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ H^1(k,G) & \longrightarrow & \prod_\mu H^1(k_\mu, G) \end{array}$$

Following the obstructions from functors, $X(k)$ is contained in the subset $X(A)^f \subset X(A)$ consisting of points of $X(A)$ whose image in $\prod_\mu H^1(k_\mu, G)$ comes from $H^1(k,G)$. One can also show that

$$X(A)^f = \bigcup_{\tau \in H^1(k,G)} f^\tau(Z^\tau(A)),$$

and that $X(A)^f$ is closed in $X(A)$ if $X$is proper. One can constrain the possible locations of rational points further by using many torsors:

$$X(A)^{H^1(X,G)} = \bigcap_{G-torsors\, f:Z\to X} X(A)^f, \quad X(A)^{descent} = \bigcap_{smooth\, affine\, G} X(A)^{H^1(X,G)}$$

Then $X(k) \subset X(A)^{descent} \subset X(A)$. One says that there is a *descent obstruction to the local-global principle* if $X(A) \neq \varnothing$ but $X(A)^{descent} = \varnothing$.

The following definition is extracted from [2], section 5.12.

**Definition 10.** 1) Let $k$ be a field, let $G$ be a smooth algebraic group over $k$. The trivial $G$-torsor over $k$, which for convenience we denote by $G$, is the underlying variety of $G$ equipped with the right action of $G$ by translation.

2) A $G$-torsor over $k$ (also called torsor under $G$ or principal homogeneous space of $G$) is a $k$ variety equipped with a right action of $G$ such that $X_{k_s}$ *equipped with its right $G_{k_s}$-action is isomorphic to $G_{k_s}$* (the isomorphism is required to respect the right actions of $G_{k_s}$). A morphism of $G$-torsors is a $G$-equivariant morphism of $k$-schemes.

3) Let $X$ be a quasi-projective $k$-variety, let $k'/k$ be a Galois extension of fields. A $k'/k$-twist (or $k'/k$-form) of $X$ is a $k$-variety $Y$ such that there exists an isomorphism $\phi : X_{k'} \to Y_{k'}$. A twist of $X$ is a $k_s/k$-twist of $X$.

4) Let $\mathcal{C}$ be a category with finite products. Then a group object in $\mathcal{C}$ is an object $G$ equipped with morphisms $m, i, e$ satisfying the group axioms: associativity, identity and inverse. A group scheme $G$ over a scheme $S$ is a group object in the category of $S$-schemes.

5) Let $G$ be a group, and let $S$ be a scheme. For each $\sigma \in G$, let $S_\sigma$ be a copy of $S$. Then $\bigsqcup_{\sigma\in G} S_\sigma$ can be made a group scheme over $S$, by letting $m$ map $S_\sigma \times_S S_\tau$ isomorphically to $S_{\sigma\tau}$ for each $\sigma, \tau \in G$. This is called a constant group scheme.

**Example 1.** 1) Let $L \supset k$ be a finite Galois extension of fields. Let $G$ be the constant group scheme over $k$ associated to $\mathrm{Gal}(L/k)$. Then the left action of $\mathrm{Gal}(L/k)$ on $L$ induces a right action of $G$ on $\mathrm{Spec}L$ that makes $\mathrm{Spec}L$ a $G$-torsor over $k$.

2) Let $T$ be the torus $x^2 + 2y^2 = 1$, then the affine variety $X$ defined by $x^2 + 2y^2 = -3$ in $\mathbb{Q}^2$ can be viewed as a $T$-torsor over $\mathbb{Q}$. It is a nontrivial torsor, since $X(\mathbb{Q}) = \varnothing$.

3) For any fixed smooth algebraic group $G$ over $k$, we have bijections:

$$\{G\text{-torsors over } k\} = \{\text{twists of } G\} = H^1(k,G)$$

The set of $k$-isomorphism classes of $k'/k$-twists of $X$ is a pointed set, with neutral element given by the isomorphism class of $X$. The action of $\mathrm{Gal}(k'/k)$ on $k'$ induces an action of $\mathrm{Gal}(k'/k)$ on the automorphism group $\mathrm{Aut}X_{k'}$. There is a natural bijection of pointed sets

$$\frac{\{k'/k\text{-twists of } X\}}{k\text{-isomorphism}} \to H^1(\mathrm{Gal}(k'/k), \mathrm{Aut}X_{k'})$$

**Proposition 8.** 1) Twisting a right torsor $f : Y \to X$ under $G$ by a cocycle $\sigma \in Z^(k, G)$ produces a right torsor $f^\sigma : Y^\sigma \to X$ under the twisted group $G^\sigma$. The subset $f^\sigma(Y^\sigma(k)) \subset X(k)$ depends only on the class $[\sigma] \in H^1(k, G)$.

2) $H^1(X, G)$ classifies $X$ torsors under $G$ up to isomorphism. We have the following partition:

$$X(k) = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(k))$$

3) Let $f : Z \to X$ be an $G$-torsor over $X$, and let $\xi$ be its class in $H^1(X, G)$. If $x \in X(k)$, then its fiber $Z_x \to \{x\}$ is a $G$-torsor over $k$, and its class in $H^1(k, G)$ will be denoted $\xi(x)$. Thus the torsor $f$ gives rise to an evaluation map

$$X(k) \to H^1(k, G), \quad x \mapsto \xi(x).$$

**Example 2.** Suppose that we want to find the rational solutions to

$$y^2 = (x^2 + 1)(x^4 + 1)$$

Write $x = \frac{X}{Z}$, where $X, Z$ are integers with $\gcd(X, Z) = 1$. Then $y = \frac{Y}{Z^3}$ for some integer $Y$ with $\gcd(Y, Z) = 1$. We get

$$Y^2 = (X^2 + Z^2)(X^4 + Z^4)$$

If a prime $p$ divides both $X^2 + Z^2$ and $X^4 + Z^4$, then

$$Z^2 \equiv -X^2 \mod p, \quad Z^4 \equiv -X^4 \mod p,$$
$$2Z^4 = (Z^2)^2 + Z^4 \equiv (-X^2)^2 + (-X^4) = 0 \mod p,$$
$$2X^4 = (X^2)^2 + X^4 \equiv (-Z^2)^2 + (-Z^4) = 0 \mod p,$$

But $\gcd(X, Z) = 1$, so this forces $p = 2$. Each odd prime $p$ divides at most one of $X^2 + Z^2$ and $X^4 + Z^4$, but the product $(X^2 + Z^2)(X^4 + Z^4)$ is a square, so the exponent of $p$ in each must be even. In other words, $X^4 + Z^4 = cW^2$ for some $c \in \{1, 2\}$. Dividing by $Z^4$ and setting $w = \frac{W}{Z^2}$, we obtain a rational point on onw of the following smooth curves:

$$Y_1 : w^2 = x^4 + 1, \quad Y_2 : 2w^2 = x^4 + 1.$$

Each curve $Y_c$ is of geometric genus $g$ where $2g + 2 = 4$, i.e. $g = 1$. The point $(x, w) = (0, 1)$ belongs to $Y_1(\mathbb{Q})$, and $(1, 1)$ belongs to $Y_2(\mathbb{Q})$, so both $Y_1, Y_2$ are open subsets of elliptic curves.

## 4.4 Rational points on $X_\epsilon$ for $\epsilon = \frac{1}{2}$

Observe that the topological structure of $X_\epsilon(\mathbb{R})$ is most simple when $\epsilon > 0$, comparing with those cases when $\epsilon = 0$ and $\epsilon < 0$. This observation inspires us to ask the following question:

**Question 1.** 1) For what $\epsilon \in \mathbb{Q}_+$, does there exist rational points on $X_\epsilon(\mathbb{R})$?

2) How many rational points are there in $X_\epsilon(\mathbb{R})$ for a given special value of $\epsilon \in \mathbb{Q}_+$?

The second observation is that we may exploit the symmetry from the certain given arrangement of the four points in problem 1. If we set $x = y = \frac{1}{2}, z = 1$, then the absolute value of $r_1, r_2, r_3, r_4$ are all equal. More specifically, we may choose $\epsilon = \frac{1}{2}$, then $p_0(\frac{1}{2} : \frac{1}{2} : 1 : 1 : 1 : 1 : 1) \in X_\epsilon(\mathbb{Q})$ is a rational point. We want to compute the tangent plane of $X_\epsilon(\mathbb{R})$ at point $p_0$, and we set $z = 1$ for convenience.

**Definition 11.** We start by naming the polynomials that generates the ideal $I(X_\epsilon)$. Let

$$\begin{aligned} F_1 &= x^2 + y^2 + \epsilon - r_1^2, & F_2 &= (x-1)^2 + y^2 + \epsilon - r_2^2, \\ F_3 &= x^2 + (y-1)^2 + \epsilon - r_3^2, & F_4 &= (x-1)^2 + (y-1)^2 + \epsilon - r_4^2. \end{aligned} \tag{11}$$

The differential of the polynomials above are given by:

$$dF_i = \frac{\partial F_i}{\partial x}dx + \frac{\partial F_i}{\partial y}dy + \frac{\partial F_i}{\partial r_1}dr_1 + \frac{\partial F_i}{\partial r_2}dr_2 + \frac{\partial F_i}{\partial r_3}dr_3 + \frac{\partial F_i}{\partial r_4}dr_4$$

The gradients, i.e. all the partial derivatives of $F_1, F_2, F_3, F_4$ are given as follows:

$$\begin{aligned} \mathrm{grad}(F_1) &= (2x, 2y, -2r_1, 0, 0, 0), \\ \mathrm{grad}(F_2) &= (2(x-1), 2y, 0, -2r_2, 0, 0), \\ \mathrm{grad}(F_3) &= (2x, 2(y-1), 0, 0, -2r_3, 0), \\ \mathrm{grad}(F_4) &= (2(x-1), 2(y-1), 0, 0, 0, -2r_4). \end{aligned} \tag{12}$$

In the chart $\{z \neq 0\}$ we set $z = 1$. Since $f_\epsilon$ gives a local parametrization of $X_\epsilon(\mathbb{R})$ near point $p_0(\frac{1}{2} : \frac{1}{2} : 1 : 1 : 1 : 1 : 1)$, whose exact expression is:

$$\begin{aligned} r_1 &= \sqrt{x^2 + y^2 + \epsilon}, & r_2 &= \sqrt{(x-1)^2 + y^2 + \epsilon}, \\ r_3 &= \sqrt{x^2 + (y-1)^2 + \epsilon}, & r_4 &= \sqrt{(x-1)^2 + y^2 + \epsilon}. \end{aligned} \tag{13}$$

The tangent vectors of $X_\epsilon(\mathbb{R})$ at any $p(x : y : r_1 : r_2 : r_3 : r_4 : 1) \in X_\epsilon(\mathbb{R})$ is given by:

$$\begin{aligned} &v_x(1, 0, \frac{x}{\sqrt{x^2+y^2+\epsilon}}, \frac{x-1}{\sqrt{(x-1)^2+y^2+\epsilon}}, \frac{x}{\sqrt{x^2+(y-1)^2+\epsilon}}, \frac{x-1}{\sqrt{(x-1)^2+(y-1)^2+\epsilon}}), \\ &v_y(0, 1, \frac{y}{\sqrt{x^2+y^2+\epsilon}}, \frac{y}{\sqrt{(x-1)^2+y^2+\epsilon}}, \frac{y-1}{\sqrt{x^2+(y-1)^2+\epsilon}}, \frac{y-1}{\sqrt{(x-1)^2+(y-1)^2+\epsilon}}). \end{aligned} \tag{14}$$

More specifically, at $p_0(\frac{1}{2} : \frac{1}{2} : 1 : 1 : 1 : 1 : 1)$, these two vectors are:

$$v_x(1, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), \quad v_y(0, 1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}).$$

We can map the variety $X$ to a hypersurface in $Z \in \mathbb{P}^3$. The morphism is given by

$$\pi : X \to \mathbb{P}^3, p(x : y : r_1 : r_2 : r_3 : r_4 : z) \mapsto \pi(p) = (x : y : r : z), \quad r = r_1 + r_2 + r_3 + r_4.$$

Define the image $\pi(X) = Z$, it is a hypersurface in $\mathbb{P}^3$ defined by the ideal

$$I(Z) =< h >, \quad h = \prod_{g \in G}(r - \sum_{1 \leq i \leq 4} g(r_1, r_2, r_3, r_4))$$

By Galois theory, $h$ is a 16-degree homogeneous polynomial in $\mathbb{Z}[x, y, r, z]$. I used SageMath to calculate its exact form, but it is too complicated to be presented here.

### 4.5 Yuan Yang's argument on rational points with fixed $x$-coordinate

**Theorem 8.** Let $r$ be a non-zero rational number, $q_1(0,0), q_3(0,1)$, the following are equivalent:
1) There is a rational point $q(x,y)$ on the line $x = r$ such that $r_1, r_3$ are rational numbers.
2) The elliptic curve $E_r : y^2 = x^3 + (\frac{1}{r^2} - 1)x^2 - \frac{2}{r^2}x + \frac{1}{r^2}$ has rank at least 1.

*Proof.* □

In [5], Berry wrote the equations in the following form with $x, y$ eliminated:

$$2(r_1^4 + z^4) + r_2^4 + r_3^4 = 2(r_2^2 + r_3^2)(r_1^2 + z^2), \quad r_1^2 + r_4^2 = r_2^2 + r_3^2.$$

The elimination of $x, y$ can be performed as follows:

$$2xz - z^2 = r_1^2 - r_2^2, \quad x = \frac{r_1^2 - r_2^2 + z^2}{2z}, \quad 2yz - z^2 = r_1^2 - r_3^2, \quad y = \frac{r_1^2 - r_3^2 + z^2}{2z},$$
$$r_1^2 = x^2 + y^2 = \frac{(r_1^2 - r_2^2 + z^2)^2}{4z^2} + \frac{(r_1^2 - r_3^2 + z^2)^2}{4z^2} = \frac{2r_1^4 + r_2^4 + r_3^4 + 2z^4 + 4r_1^2z^2 - 2(r_1^2 + z^2)(r_2^2 + r_3^2)}{4z^2},$$
$$2r_1^4 + r_2^4 + r_3^4 + 2z^4 = 2(r_1^2 + z^2)(r_2^2 + r_3^2)$$

The quartic equation is the only constraint for 3-distance problem of $q_1(0,0), q_2(1,0), q_3(0,1)$. It turns out that its complex points $S(\mathbb{C})$ is a Kummer surface, i.e. a quartic surface with exactly 16 singular points. Actually in the 3-distance problem, there are infinitely many one-parameter families of rational points on $S$. In [5], the author summed up our knowledge of the 3-distance problem in the following theorem:

**Theorem 9.** 1) The curves given in the table below are parametrized curves on $S$. Up to symmetry, there are no further parametrizable cueves on $S$ of degrees 2 or 4.

2) Infinitely many parametrizable curves can be obtained starting from $C_2$, by successive projections from nodes 1 and 2.

$$r_1 = 1 - t^2, \quad r_2 = t^2 + 2t - 1, \quad r_3 = t^2 + 1, \quad z = r_1 + r_2,$$
$$r_1 = 4t^4 - 48t^3 + 192t^2 - 384t + 256, \quad r_2 = 5t^4 - 48t^3 + 144t^2 - 128t + 64,$$
$$r_3 = t^4 - 16t^3 + 144t62 - 384t + 320, \quad z = \frac{r_1 + r_3}{2},$$
$$r_1 = t^6 + 12t^5 + 21t^4 - 16t^3 - 21t^2 + 4t - 1, \quad r_2 = t^6 + 6t^5 + 29t^4 + 44t^3 - 13t^2 - 2t - 1,$$
$$r_3 = t^6 + 4t^5 + 7t^4 + 24t^3 + 39t^2 - 12t + 1, \quad z = 10t^5 + 40t^4 + 28t^3 - 24t^2 + 10t,$$
$$r_1 = 8t^7 - 16t^6 - 8t^5 - 8t^3 + 16t^2 + 8t, \quad r_2 = t^8 - 8t^7 + 12t^6 + 24t^5 - 10t^4 - 24t^3 + 12t^2 + 8t + 1,$$
$$r_3 = t^8 + 12t^6 - 32t^5 - 10t^4 + 32t^3 + 12t^2 + 1, \quad z = t^8 4t^6 + 22t^4 - 4t^2 + 1,$$

While solving the equation in $\mathbb{Q}_p^{dist}$ for sufficiently large $p$, can we identify which point has small height? If $p \in X(\mathbb{Q}_p^{dist})$ is such a point that $x_p, y_p \in \mathbb{Q}$, with their heights bounded by $H$, what can we say about it? We require that $p$ is larger than height of $x^2 + y^2, (x-1)^2 + y^2, x^2 + (y-1)^2, (x-1)^2 + (y-1)^2$. A well-known theorem is that there are only finitely many extensions of $\mathbb{Q}_p$ having a fixed degree. When the degree is 2, quadratic extensions of $\mathbb{Q}_p$ are classified by $(\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2)\backslash\{1\}$. When $p \neq 2$, $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 = \{1, \omega, p, \omega p\}$ where $\omega$ is a non-quadratic residue modulo $p$. When $p = 2$,

$\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 = \{1,3,5,7,2,6,10,14\}$. Let $r_{1,0} \in [p] \cup [p]\sqrt{\omega}$ be a solution to $x^2+y^2 \equiv r_{1,0}^2 \mod p$. Then the analytical solution is

$$r_1 = (x^2+y^2)^{1/2} = r_{1,0}(1+(\frac{x^2+y^2}{r_{1,0}^2}-1))^{1/2} = r_{1,0}\sum_{k\geq 0}\binom{1/2}{k}t_1^k, \quad t_1 = \frac{x^2+y^2}{r_{1,0}^2}-1.$$

Similarly we can define the following variables:

$$r_2 = ((x-1)^2+y^2)^{1/2} = r_{2,0}(1+(\frac{(x-1)^2+y^2}{r_{2,0}^2}-1))^{1/2} = r_{2,0}\sum_{k\geq 0}\binom{1/2}{k}t_2^k,$$

$$r_3 = (x^2+(y-1)^2)^{1/2} = r_{3,0}(1+(\frac{x^2+(y-1)^2}{r_{3,0}^2}-1))^{1/2} = r_{3,0}\sum_{k\geq 0}\binom{1/2}{k}t_3^k,$$

$$r_4 = ((x-1)^2+(y-1)^2)^{1/2} = r_{4,0}(1+(\frac{(x-1)^2+(y-1)^2}{r_{4,0}^2}-1))^{1/2} = r_{4,0}\sum_{k\geq 0}\binom{1/2}{k}t_4^k,$$

$$t_2 = \frac{(x-1)^2+y^2}{r_{2,0}^2}-1, \quad t_3 = \frac{x^2+(y-1)^2}{r_{3,0}^2}-1, \quad t_4 = \frac{(x-1)^2+(y-1)^2}{r_{4,0}^2}-1.$$

In order to prove that $r_1, r_2, r_3, r_4$ aren't all rational, it suffices to show that for any $x, y \in \mathbb{Q}$, at least one of $r_1, r_2, r_3, r_4$ has bad rational approximation. That is, $|r_i - r'|_p > \delta$ for any rational $r'$ with bounded height. We formalize this idea as follows:

$$|\sqrt{x^2+y^2}-r_1'|+|\sqrt{(x-1)^2+y^2}-r_2'|+|\sqrt{x^2+(y-1)^2}-r_3'|+|\sqrt{(x-1)^2+(y-1)^2}-r_4'| > \delta(H) > 0.$$

for a suitably chosen norm $|\cdot|$, and $x, y, r_1', r_2', r_3', r_4' \in \mathbb{Q}$ are rationals with bounded height $H$. Note that square root of $a$ over $\mathbb{R}$ can be calculated by the iteration $x_{n+1} = \frac{x_n}{2} + \frac{a}{2x_n}$ starting from $x_0 = 1$. This is known as Babylonian method, it's a quadratically convergent algorithm. We may also use continued fractions to approach $r_i$. If $r_i \in \mathbb{Q}$, then its continued fraction terminates. The problem turns to the following question: for any $x, y \in \mathbb{Q}$, prove that at least one of the 4 continued fractions is infinite.

$$|x^2+y^2-u_1^2| + |(x-z)^2+y^2-u_2^2| + |x^2+(y-z)^2-u_3^2| + |(x-z)^2+(y-z)^2-u_4^2| > 0.$$

We want the inequality above hold for any integers with $x, y, z$ not identically zero. The only possibilities of $(x, y, z)$ modulo 2 are $(1,0,0), (0,1,0)$. The only possibilities of $(x, y, z)$ modulo 4 are $(1,0,0), (3,0,0), (0,1,0), (0,3,0)$. But I think it's not a good idea to formulate the problem in terms of $\mathbb{Z}$ coefficients inequality, because it's hard to solve it, just as in the case of integer programming. The only way to solve it in this way uses diophantine approximation.

# Appendix

# References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[5] T.G. Berry Points at rational distance from the corners of a unit square.